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SOME SPECIAL METHODS OF HOMEOMORPHISM THEORY IN INFINITE-DIMENSIONAL TOPOLOGY

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1. Introduction.

In studying the topology of infinite-dimensional spaces which are of particular interest in analysis and algebra, it has been customary to treat the spaces, where possible, as topological linear spaces and to use methods involving linear analysis and convexity. In this report we embed certain topological linear spaces in compact spaces and consider special methods applicable to compact spaces, particularly infinite product spaces. We study not only the establishment of homeomorphisms between spaces but also extensions of existing homeomorphisms defined on closed subsets.

Our principal specific infinite-dimensional spaces will be (1) Hilbert space, l_2 (the space of square summable sequences with the norm topology), (2) The countable infinite product of lines, s, and (3) The Hilbert cube or parallelotope, I^∞, (the countable infinite product of closed intervals).

Within the past year, results of Kadec [8] together with results of the author [1] and based on previous results of Bessaga and Pelczynski [4], [6] have established that all infinite-dimensional separable Fréchet spaces are homeomorphic (where a Fréchet space is defined as a complete, locally convex linear metric space). Specifically, Bessaga and Pelczynski showed "Under the conjecture that all separable infinite-dimensional Banach spaces are homeomorphic with l_2, every separable infinite-dimensional Fréchet space X, with $X \neq s$, is homeomorphic with l_2", Kadec showed that all separable infinite-dimensional Banach spaces are homeomorphic with l_2 and the author showed that l_2 is homeomorphic with s. In this paper we shall give an outline of a proof of this last result which, incidentally, settles a specific question raised by Fréchet [7] in 1928.

The complete topological classification of infinite-dimensional separable Fréchet spaces means that many further homeomorphism questions concerning such spaces are reduced to homeomorphism questions concerning any model of such spaces. Of course, one such model is l_2. In a number of papers [10], [11] etc., Klee and others established numerous topological properties of l_2 and its subsets by methods involving linear analysis and convexity. In this paper we shall be primarily concerned with s as our model and specifically with s embedded canonically in the Hilbert cube.

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In fact, the proof that l_2 is homeomorphic to s follows more from considerations of the topology of s than from those of l_2.

For each $j > 0$, let I_j be the closed interval $[0, 1/2^j]$ and let I_j^0 be the open interval $(0, 1/2^j)$.

We let the symbol "$\sim$" mean "is homeomorphic to". Then $I^\infty \sim \prod_{j>0} I_j$ and $s \sim \sim \prod_{j>0} I_j^0$. For simplicity, we consider $I^\infty = \prod_{j>0} I_j$ and $s = \prod_{j>0} I_j^0$ with the metric of either defined by $d(p, q) = \sqrt{(\sum(p_i - q_i)^2)}$, $p = \{p_i\}$, $q = \{q_i\} \in I^\infty$ or s. Clearly, $s \subset I^\infty$, and both s and $I^\infty \setminus s$ are dense in I^∞. Also neither s nor $I^\infty \setminus s$ is locally compact. We denote $I^\infty \setminus s$ as $B(I^\infty)$. Notice that in this version of s, s is not a topological linear space as such and s fails to be complete but many properties of s are deducible from properties of I^∞.

For any $i > 0$, let τ_i denote the projection of I^∞ on I_i. For any $i > 0$, either set $\tau_i^{-1}(0)$ or $\tau_i^{-1}(1/2^i)$ is called an *endslice* of I^∞.

In dealing with questions of homeomorphisms of infinite-dimensional spaces, we have available the usual theorems for determining that two spaces (or subspaces) are not homeomorphic if, in fact, they are not. For example, we have homotopy properties and local homotopy properties, compactness and local compactness, σ-compactness and topological completeness. We note that s and $B(I^\infty)$ are not homeomorphic since $B(I^\infty)$ is clearly σ-compact (the countable union of compact sets, in this case, endslices) and s cannot be σ-compact since no compact set in s contains an open subset of s and s is a G_δ subset of I^∞.

In studying homeomorphisms of I^∞ onto itself, it seems to be particularly useful to consider I^∞ as $s \cup B(I^\infty)$. In many cases, valuable distinctions can be made on the basis of whether a particular subset of I^∞ is in s, is in $B(I^\infty)$, or intersects both.

Definitions. A homeomorphism h of I^∞ onto itself is said to be a *β-homeomorphism* if $h(B(I^\infty)) \subset B(I^\infty)$ and to be a *β^*-homeomorphism* if $h(B(I^\infty)) = B(I^\infty)$.

It frequently is easy to decide whether a particular homeomorphism which we have constructed is β (or β^*) since a point p of I^∞ is in $B(I^\infty)$ iff for some i, $\tau_i(p) \in \in I_i \setminus I_i^0$. Thus we may keep track of points since both s and $B(I^\infty)$ are easily identifiable.

Clearly, any β^*-homeomorphism induces a homeomorphism of s onto itself and any β-homeomorphism induces a homeomorphism of s onto a subset of I^∞ containing s.

In establishing homeomorphisms by limiting processes it is, of course, highly useful for the space concerned to be complete and in some contexts even more useful for the space to be compact. The study of homeomorphisms of s as restrictions of homeomorphisms of I^∞ onto itself seems to the author to be substantially more fruitful than studies of homeomorphisms of s itself.

Definitions. A subset K of I^∞ is *partially deficient in the j-th direction* if $\tau_j(K)$

is contained in a closed interval in I_j^0 and is *deficient in the j-th direction* if $\tau_j(K)$ consists of a single point in I_j^0. A subset K of I^∞ is *of infinite (partial) deficiency* if it is (partially) deficient in each of infinitely many directions. Deficiency in s or l_2 may be similarly defined.

The concept of infinite deficiency plays a vital role in the topology of all three of these spaces as is well known for s and l_2 and is illustrated for I^∞ by several of the lemmas and theorems to follow.

In the development of topological and geometric intuition about spaces such as I^∞ and s and of homeomorphisms of these spaces, there are several considerations which the author feels are helpful. They are listed from (A) to (G) below.

(A) For any sequence $\{f_i\}$ of homeomorphisms of I^∞ onto itself, the sequence $\{f_i \circ \ldots \circ f_2 \circ f_1\}_{i>0}$ converges to a homeomorphism of I^∞ onto itself provided that the sequence of distances of the f_i from the identity converges to zero sufficiently rapidly. This proposition follows easily by a straightforward argument involving uniform continuity or by considerations like those for the Baire Category Theorem in the space of homeomorphisms of I^∞ onto itself. This proposition is extremely useful in infinite-dimensional product space topology since in many inductive constructions, the homeomorphism f_i may be chosen to be as close to the identity as we wish.

(B) For any $\varepsilon > 0$ there are only finitely many coordinate spaces of I^∞ or s which are of diameter greater than ε. This fact is very useful in applying (A).

(C) I^∞ is homogeneous, that is, any point can be carried onto any other point by a homeomorphism of I^∞ onto itself. This result was first proved by Keller [9]. We give here an almost trivial proof using (A) and (B) at the critical step. First, for $p = \{p_i\}$ and $q = \{q_i\}$ with $p, q \in s$, for each i, let h_i be a homeomorphism of I_i onto itself such that $h_i(p_i) = q_i$. Then h defined coordinatewise, for each i, as h_i on I_i is a β^*-homeomorphism carrying p onto q. Now it suffices to observe that for any particular $p \in B(I^\infty)$, there exists a homeomorphism f of I^∞ onto itself such that $f(p) \in s$. We define f as $\lim_{i\to\infty} \{f_i \circ \ldots \circ f_2 \circ f_1\}$ with f_i defined inductively. Each f_i is to affect only two coordinates I_i and I_{n_i} and $\tau_i[f_i(f_{i-1} \circ \ldots \circ f_2 \circ f_1(p))]$ is to be an element of I_i^0. Furthermore, the n_i is to be so chosen that (1) $n_i > i$ and (2) I_{n_i} is so small that f_i can satisfy the implied convergence criterion of (A).

The homogeneity of I^∞ together with the fact that both s and $B(I^\infty)$ are dense in I^∞ means that $B(I^\infty)$ plays a very different rôle than does the boundary of a finite-dimensional cell. In fact, it can be shown [3] that there is a homeomorphism h of I^∞ onto itself such that $h[B(I^\infty)] \subset s$.

(D) I^∞, s, and l_2 all satisfy a very strong "non-invariance of domain" property. Each space contains many closed nowhere dense topological copies of itself. In this sense, such infinite-dimensional spaces are strongly non-Euclidean in character and much more like the Cantor Set or the Universal Curve than like Euclidean spaces.

(E) Many processes involving a single set have immediate analogues involving countable collections of sets. This is particularly true for sets of infinite deficiency. See Lemmas (1) and (2) of Section 2.

(F) In some cases phenomena can be "passed off to infinity" by successive interchange of coordinates. This method is effectively used by Raymond Wong [13] in proving that every homeomorphism of s, l_2 or I^∞ onto itself is isotopic to the identity. In a certain sense, it is used in the arguments for many of the lemmas and theorems cited below and was used in (C) in the short proof of the homogeneity of I^∞.

(G) As indicated by the result previously given concerning Fréchet spaces and as substantiated by earlier results of Klee and by results to be cited in this paper, many surprisingly strong homeomorphism theorems can be proved concerning l_2, s, and I^∞. Other examples include the isotopy results of Wong mentioned in (F) and the somewhat stronger results that each homeomorphism of l_2, s or I^∞ onto itself is stable in the sense of Brown and Gluck (due to Wong for l_2 and s and the author for I^∞). Analogous isotopy and stability questions are still open for orientation-preserving homemorphisms of an n-sphere or n-cell onto itself.

2. Establishing Homeomorphisms.

The following three lemmas represent processes which are very useful in homeomorphism theory in I^∞ and s. The actual lemmas used in a particular case may involve slight modifications in the direction of controlling extra conditions such as the way the homeomorphisms concerned affect particular coordinates. *For each of these three lemmas the homeomorphisms asserted to exist can also be required to be arbitrarily close to the identity.* Thus the lemmas are applicable with respect to the convergence criterion (A) of the previous section.

Observe that each of Lemmas 1 and 2 is stated for a countable collection of sets. The argument for each is only a routine modification of a conceptually easier argument involving just a single set.

Lemma 1. *For any countable collection of closed sets each of infinite partial deficiency in I^∞ there is a β^*-homeomorphism carrying each onto a set of infinite deficiency.*

The proof of Lemma 1 (see [2]) is not difficult using (A) and (B) of Section 1 and a local "tilting" process.

Lemma 2. *For any countable collection $\{K_i\}$ of closed sets of infinite deficiency in I^∞, there is a β-homeomorphism h carrying each onto a set of infinite deficiency in some endslice such that $h(s \setminus \bigcup K_i) = s$.*

The proof of Lemma 2 (see [2] and [3]) involves an infinite twisting procedure and is a straightforward modification of an argument for "pushing" exactly one point off s to $B(I^\infty)$.

Lemma 3. *For any endslice W, there exists a β^*-homeomorphism h such that $h(W) \subset W$ and $h(W)$ intersects no other endslice.*

Lemma 3 may be proved (see [2]) by a straightforward argument.

It is particularly useful in relating Lemmas 1 and 2 to the lemmas of the next section.

We now give several theorems whose proofs depend heavily on the procedures represented by the foregoing lemmas.

Theorem 1. *For any separable metric space X and σ-compact subset K of $(X \times s)$, $(X \times s) \sim (X \times s) \setminus K$.*

This theorem has some useful corollaries.

Corollary 1. *For any σ-compact subset K of s, $s \sim s \setminus K$.*

The proof of Theorem 1 is but a slight modification of a direct (independent) proof of Corollary 1. And Corollary 1 follows immediately from Lemmas 1 and 2.

To obtain a second corollary of Theorem 1 we employ a theorem of Bessaga and Pelczynski ([4] or [6]) that $(l_2 \times s) \sim l_2$.

Corollary 2. *For any σ-compact subset K of l_2, $l_2 \sim (l_2 \setminus K)$.*

We are now in a position to outline a proof of the theorem that $s \sim l_2$.

Theorem 2. $s \sim l_2$.

Outline of proof: Step 1. Let $\tilde{l}_2$ be the set of all points p of l_2 such that $\|p\| = 1$ and p has infinitely many non-zero coordinates. By use of Corollary 2 it can easily be shown that $l_2 \sim \tilde{l}_2$.

Step 2. For $\{x_i\} \in \tilde{l}_2$ and $\{y_i\} \in s$, let f be a function of $\tilde{l}_2$ into s defined in terms of coordinates by

$$y_i = \frac{1}{2^{i+1}} \cdot \frac{x_i + \sqrt{\left(1 - \sum_{j=1}^{i-1} x_j^2\right)}}{\sqrt{\left(1 - \sum_{j=1}^{i-1} x_j^2\right)}}.$$

Then f is a homeomorphism of $\tilde{l}_2$ into s as may be verified. The intuition for f comes from an attempt to embed $\tilde{l}_2$ densely in s. The geometry leading to f comes from the following: Let S_i be the set of all non-degenerate intersections of $\mathrm{Cl}(\tilde{l}_2)$ with hyperplanes of deficiency i orthogonal to the i-dimensional hyperplane spanned by the first i coordinate axes. Let f_i be the "natural" inductively defined $1-1$ function of S_i onto $I_1^0 \times I_2^0 \times \ldots \times I_i^0$. In a sense, f_{i+1} "refines" f_i. Then f is the "limit" of the f_i.

Step 3. The embedding f of $\tilde{l}_2$ into s has the (non-obvious) property that $s \setminus f(\tilde{l}_2)$ is the countable union of compact sets. Hence Corollary 1 is applicable.

Thus we have $l_2 \sim \tilde{l}_2 \sim f(\tilde{l}_2) \sim s$.

The proof of the following theorem uses a classical result of Sierpinski [12] concerning G_δ's together with some elementary geometry of n-cells and repeated use of the procedures of Lemmas 1 and 2. First we need a definition.

Definition. For any $n > 0$, let R_n be a closed n-cell with interior R_n^0. Any set V such that $R_n^0 \subset V \subset R_n$ is called a *near n-cell.*

Theorem 3. *Let* $X = \prod_{i>0} V_i$ *where for each* i, V_i *is a near* n_i*-cell. Then* $X \sim s$ *iff each* V_i *is a* G_δ *subset of a closed* n_i*-cell and for infinitely many* i, V_i *is not a closed cell.*

As a special case of Theorem 3 we have a result obtained also and independently by Bessaga and Klee [5].

Corollary 3. *A countable infinite product of intervals (each open, half-open or closed) is homeomorphic to s iff infinitely many of its factors are open or half-open.*

3. Homeomorphism Extension Theorems.

Because of the important rôles played by sets of infinite deficiency in s, in l_2 and in I^∞ it is natural to ask for topological characterizations of such sets. Analogous results for s, l_2 and I^∞ are obtained. In fact, we shall get a characterization of such sets as a corollary of a result giving necessary and sufficient conditions under which a homeomorphism from a closed set onto a closed set of infinite deficiency can be extended to a homeomorphism of the space onto itself. The details are contained in [3].

The results seem surprisingly simple and categorical. The conditions are independent of both the particular homeomorphism and the topology of the set itself. They depend only on the homotopy properties (global and local) of the complement of the set.

Definition. A closed set K in I^∞ (or s or l_2) has *Property Z* if for each non-null homotopically trivial open set U in I^∞ (or s or l_2), $U \setminus K$ is non-null and homotopically trivial.

A lemma of the following type is fundamental in homeomorphism extension theory. Its proof is by a standard method of Klee [7].

Lemma 4. (*Klee's Extension Lemma*). *Any homeomorphism of a compact subset K of s of infinite deficiency into a subset of s of infinite deficiency can be extended to a* β^**-homeomorphism.*

This lemma is proved by putting the compact set K and its image into complementary infinite-dimensional subspaces and mapping from "K" to its graph and from the graph to the "image of K". Under the altered hypotheses that K and its image merely be closed subsets of infinite deficiency we may conclude that there exists an extension which is a homeomorphism of s onto itself.

The final lemma to be cited translates Property Z and homotopy properties into homeomorphism properties.

Lemma 5. *For any $\varepsilon > 0$ and any closed set K with Property Z and any countable collection $\{M_i\}$ of finite polyhedrons in s there is a β^*-homeomorphism h such that (1) $h(K) \cap (\bigcup M_i) = \emptyset$ and (2) h is within ε of the identity.*

Note that the ε-condition enables this lemma to be used with the convergence criterion of (A). The proof of this lemma is relatively complicated and involves using a preliminary weaker form of the extension theorem given below. This lemma is used in getting from a weak form of the extension theorem to its full strength.

We now state the homeomorphism extension theorem. Observe that the same conditions apply for the compact space I^∞ as for s and l_2.

Theorem 4. *In order that a homeomorphism from a closed set K in I^∞ (or in s or in l_2) onto a closed set of infinite deficiency can be extended to a homeomorphism of I^∞ (or s or l_2) onto itself it is necessary and sufficient that K have Property Z.*

The proof of this theorem is rather lengthy and depends on the convergence property (A) and the procedures of all the lemmas cited previously. Although this theorem is applicable to l_2, its conjecture and proof depended primarily upon considerations in s and I^∞.

Since there do exist closed sets of infinite deficiency which are homeomorphic to the whole space (I^∞, s, or l_2) we have the following corollary.

Corollary 4. *Given a closed set K in I^∞ (or in s or in l_2). In order that there exist a homeomorphism of I^∞ (or s or l_2) onto itself carrying K into a set of infinite deficiency, it is necessary and sufficient that K have Property Z.*

Among sets with Property Z are

(1) Any compact subset of s or l_2.
(2) Any closed set which is the countable union of sets of infinite deficiency.
(3) Any finite union of endslices of I^∞ (none has infinite deficiency).
(4) Any compact subset of $B(I^\infty)$.

There are, of course, many closed sets which do not have Property Z. It has recently been shown by Wong [13] that there exist "wild" closed zero-dimensional subsets of I^∞, s and l_2. Their complements are not simply connected and thus the sets do not have Property Z.

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