PROJECTION-SPECTRA

P. ALEXANDROFF and V. PONOMAREV

Moskva

1. Definition of the projection spectrum and its limit space.

A projection spectrum is an indexed family of simplicial complexes[1]) $\mathcal{N}_\alpha$ where the indices $\alpha, \alpha', \ldots$ form a directed set $A = \{\alpha\}$ and for any pair α, α' with $\alpha' \succ \alpha$ in A a simplicial map

$$\pi_\alpha^{\alpha'} : \mathcal{N}_{\alpha'} \to \mathcal{N}_\alpha$$

of $\mathcal{N}_{\alpha'}$ onto $\mathcal{N}_\alpha$ is defined such that the transitive law

$$\pi_\alpha^{\alpha''} = \pi_\alpha^{\alpha'} \pi_{\alpha'}^{\alpha''} \quad \text{for any} \quad \alpha'' \succ \alpha' \succ \alpha$$

is fulfilled. The mappings $\pi_\alpha^{\alpha'}$ are called projections.

The limit space $\tilde{S}$ of the spectrum

$$S = (\mathcal{N}_\alpha, \pi_\alpha^{\alpha'}) \tag{1}$$

is defined as follows:

In each complex $\mathcal{N}_\alpha$ of the spectrum (1) take a simplex t_α in such a way that for $\alpha' \succ \alpha$ we always have

$$t_\alpha = \pi_\alpha^{\alpha'} t_\alpha .$$

The set $\xi = \{t_\alpha\}$ of these simplexes t_α is called a *thread* of the spectrum S. A thread $\xi = \{t_\alpha\}$ is called maximal if there exists no thread $\xi' = \{t'_\alpha\}$ different from ξ and such that $t'_\alpha \geqq t_\alpha$ (that means that t_α is a face of t'_α) for all α.

By definition, *the maximal threads of the spectrum S are the points of its limit space $\tilde{S}$.*

As for the topology of $\tilde{S}$, we define, for any simplex t_{α_0} of an arbitrary $\mathcal{N}_{\alpha_0}$, the set Ot_{α_0} consisting of all maximal threads $\xi = \{t'_\alpha\}$ with $t'_{\alpha_0} \leqq t_{\alpha_0}$. These sets Ot_α are by definition the basic open sets of $\tilde{S}$.

It is easily seen that $\tilde{S}$ with this topology is a T_1-space. If the spectrum (1) is finite (i.e., if all complexes $\mathcal{N}_\alpha$ are finite) then its limit space is a bicompact T_1-space.

[1]) "Complex" is meant in the classical sense as a set $\mathcal{N}$ of (abstract finite dimensional) simplexes; if $t \in \mathcal{N}$ and $t' < t$ (i.e. t' is a face of t), then $t' \in \mathcal{N}$.

These definitions (with a countable indexing set $\{\alpha\}$ and finite complexes $\mathcal{N}_\alpha$) were given by Alexandroff [1]; it was through the definition of the projection spectrum that the notion of an inverse system first entered mathematics.

It is proved in [1] that every (n-dimensional) compactum (=compact metric space) is the limit space of a (n-dimensional) projection spectrum.

A. Kurosh [3] was the first to consider arbitrary finite spectra (i.e., spectra whose complexes $\mathcal{N}_\alpha$ are finite but the indexing set $\{\alpha\}$ is an arbitrary directed set). The notion of projective spectrum in its full generality (as given at the beginning of this paper) was introduced by Ponomarev [4], [5].

2. The spectrum and the finite spectrum of a space.

A cover $\alpha = \{A_\lambda^\alpha\}$ of a space X is called canonical if α is locally finite and if its elements are canonical closed sets $A_\lambda^\alpha = [U_\lambda^\alpha]$, $U_\lambda^\alpha = \operatorname{Int} A_\lambda^\alpha$ with disjoint open kernels U_λ^α.

Let the set of all canonical coverings α of a space X be directed (in the natural manner: $\alpha' \succ \alpha$ if α' refines α), then the simplicial mappings $\pi_\alpha^{\alpha'}$ of the nerve $\mathcal{N}_{\alpha'}$ of $\alpha' \succ \alpha$ onto the nerve $\mathcal{N}_\alpha$ of α are defined in a natural way and we get the projection spectrum $S = (\mathcal{N}_\alpha, \pi_\alpha^{\alpha'})$, called *the maximal projection spectrum* (or simply *the spectrum*) SX of the space X.

If we take only the finite canonical coverings (but all of these) we get, in the same way, *the (maximal) finite spectrum* S_fX of X.

Kurosh proved in [3] that every bicompactum (= bicompact Hausdorff space) is the limit space of its spectrum.[2])

Ponomarev proved in [4], [5] that every paracompactum (= paracompact Hausdorff space) X is the limit space $\tilde{S}X$ of its spectrum SX.

Alexandroff proved in [2] that the limit space $\tilde{S}_fX$ of the finite spectrum S_fX of a normal space is the Stone-Čech extension βX of X.

For a regular and even semiregular[3]) X, the limit space $\tilde{S}_fX$ of the finite spectrum S_fX is the bicompact T_1-space $\omega^\varkappa X$ whose points are maximal centred systems of canonical closed sets and the topology is the Wallman topology. The space $\omega^\varkappa X$ was defined in [14], but the complete proof of the equality $\tilde{S}_fX = \omega^\varkappa X$ was first given by V. Zaicev [15] (the original proof given in [14] contained a gap because $\omega^\varkappa X$ need not coincide with the space ωX of Wallman).

Moreover, Zaicev [15] proved the following theorems:

1. *If X is a topological space and the finite spectrum S_fX exists* (in the sense that there are "sufficiently many" finite canonical covers), *then the limit space of S_fX is $\omega^\varkappa X$.*

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[2]) All canonical coverings of a bicompact space are finite, therefore the maximal spectrum of a bicompact space coincides with its (maximal) finite spectrum.

[3]) A T_1-space is called semi-regular or a T_ζ-space if its canonical open sets form a basis.

2. *For every regular space X the space $\omega^{\varkappa}X$ is semi-regular and a* (*bicompact*) *extension of X; thus every regular space has a semi-regular bicompact extension.*

3. *A bicompact T_1-space X is* (in a natural way) *homeomorphic to the limit space of its spectrum if and only if the space X is semi-regular.*

It follows that the spectrum of $\omega^{\varkappa}X$ is identical with the spectrum of X and therefore $\omega^{\varkappa}(\omega^{\varkappa}X) = \omega^{\varkappa}X$.

3. Strengthening and weakening of a spectrum. The absolute.

A complex B is a strengthening of a complex A (or A is a weakening of B) if both complexes have the same vertices and A is a subcomplex of B. The zero-dimensional complex $\dot{B}$ consisting of all vertices of the complex B is called the *total weakening of B.*

A projection spectrum $S' = (\mathcal{N}'_\alpha, \pi_\alpha^{\alpha'})$ is a strengthening (a weakening) of the projection spectrum $S = (\mathcal{N}_\alpha, \pi_\alpha^{\alpha'})$ if the index set $\{\alpha\}$ is the same in both spectra and each complex $\mathcal{N}'_\alpha$ is a strengthening (a weakening) of the corresponding $\mathcal{N}_\alpha$.

One of the main results of the spectral theory of irreducible perfect mappings $f : X \to Y$ of paracompacta, given by Ponomarev in [5], is that to such a mapping there corresponds a strengthening fSX of the spectrum SX of X, and that the spectrum fSX has the space Y for its limit space.

Thus to the irreducible perfect images (preimages, respectively) of a given paracompactum X there correspond well defined strengthenings (weakenings) of the spectrum SX which completely describe the given mapping f.

The total weakening $\dot{S}$ of the spectrum of X has as its limit space a zero-dimensional paracompactum $\dot{X}$ called the *absolute* of the space. This is a preimage of X under a well defined perfect mapping π_X (corresponding to the strengthening of $\dot{S}$ to SX), and this preimage is maximal in the following sense: every preimage of $\dot{X}$ under an irreducible perfect mapping φ is homeomorphic to $\dot{X}$ (and the mapping φ is necessarily topological).

If there exists an irreducible perfect mapping f of the paracompactum X onto the paracompactum Y, then the absolutes $\dot{X}$ and $\dot{Y}$ are necessarily homeomorphic, and

$$f = \pi_Y h \pi_X^{-1} \tag{2}$$

where h is a suitably chosen topological mapping of $\dot{X}$ onto $\dot{Y}$. Conversely, to each topological map $h : \dot{X} \to \dot{Y}$ there corresponds by (2) *a perfect but in general multivalued and* (in a certain sense) *still irreducible map $f : X \to Y$.*

The spectral theory of absolutes and of irreducible perfect mappings of paracompact spaces is given with all details in Ponomarev's paper [5].

4. Remarks on the history and further development of the theory of absolutes.

The first construction of the absolute of a bicompactum is already contained in the classical 1937 memoir [12] of M. H. Stone. In the fall of 1958 a paper [13] of Gleason appeared in which Stone's construction is taken up anew for bicompacta and for locally bicompact spaces; as a matter of fact, this construction appeared also in 1940 as an auxiliary tool in a paper of Fomin [9]. Gleason proved the projectivity of the absolute, and thereby in fact established the property of the absolute to induce via its homeomorphisms all the single-valued irreducible continuous (or perfect) mappings of bicompacta (or locally bicompact spaces, respectively). However, the explicit formula (2) giving all (even multivalued) irreducible perfect mappings was first stated by Ponomarev in [4], [5] (and three years later in the paper [16] of M. Henriksen and M. Jerison).

After the spectral theory of absolutes and irreducible perfect mappings was set up in [4], [5], S. Iliadis [8], 1963, gave a new approach to the subject on the basis of maximal centred systems (= ultrafilters) of open sets. Thus a new and brilliant application of an old method of Alexandroff [17], 1939, was found. Iliadis constructed his theory of absolutes for all regular and indeed for all Hausdorff spaces[4]). Immediately after the work of Iliadis there appeared a paper by Ponomarev [7] in which the same degree of generality was obtained by spectral method, and, indeed, by the method of Ponomarev [6], 1960.

Let us sketch this last method.

Take the finite spectrum

$$S \equiv S_f X = (\mathcal{N}_\alpha, \pi_\alpha^{\alpha'}) \tag{3}$$

of a (Hausdorff) space X; the complexes $\mathcal{N}_\alpha$ are nerves of the finite canonical coverings

$$\alpha = \{A_1^\alpha, \ldots, A_{s_\alpha}^\alpha\}$$

of the space X. Let

$$S^\bullet = (\mathcal{N}_\alpha^\bullet, \pi^{\alpha'}), \quad \mathcal{N}_\alpha^\bullet = \{a_1^\alpha, \ldots, a_S^\alpha\} \tag{4}$$

be the total weakening of the spectrum (3).

The limit space of $S^\bullet$ is a zerodimensional bicompactum $\overline{\mathscr{D}}$. Let $\mathscr{D}$ be the set of all points of $\overline{\mathscr{D}}$, i.e. of threads

$$\xi = \{a_{i_\alpha}^\alpha\}$$

of $S^\bullet$ such that for the $A_{i_\alpha}^\alpha \sim a_{i_\alpha}^\alpha$ one has

$$\bigcap_\alpha A_{i_\alpha}^\alpha \neq \emptyset .$$

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[4]) A rather complete exposition of results of Iliadis may be found in [1].

The set $\mathscr{D}$ is dense in $\overline{\mathscr{D}}$; as $\overline{\mathscr{D}}$ is a bicompactum, $\mathscr{D}$ is completely regular; *$\mathscr{D}$ is the absolute of the space X*; if X is paracompact, $\mathscr{D}$ coincides (up to a natural homeomorphism) with the spectral absolute $\dot{X}$ (and this is interesting in itself). In the general case of a Hausdorff space X, the space $\mathscr{D}$, which we again denote by $\dot{X}$, is homeomorphic in a natural way to the absolute obtained by Iliadis.

Furthermore, the bicompactum $\overline{\mathscr{D}}$ is the Stone-Čech extension of the absolute,

$$\overline{\mathscr{D}} = \beta\mathscr{D} .$$

The fundamental properties of the absolute – in particular, formula (2) and the property of the absolute to be an extremally disconnected space – hold in the general case; however, for a non-regular (Hausdorff) X, the natural map π_X, which is continuous (and perfect) for a regular X, is in general only θ-continuous in the sense of Fomin [9], and still perfect and irreducible.

Of course, *the absolute $\dot{X}$ is paracompact if and only if the space X is itself a paracompactum.*

5. A final remark concerning the theorems of V. Zaicev.

It was mentioned in § 2 that the space $\omega^\varkappa X$ is, for a regular X, a semi-regular bicompact T_1-extension of X (or, as we say for conciseness, "a bicompact T_ζ-extension"). Moreover, Zaicev proved that the extension $\omega^\varkappa X$ has the characteristic property of being maximal among all bicompact T_ζ-extensions of the given regular X, in the sense that the spectrum of any bicompact T_ζ-extension of X is a strengthening of the spectrum of $\omega^\varkappa X$ (or, which is the same, of the spectrum of X itself).

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