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TWO CLASSES OF ALMOST PERIODIC FUNCTIONS ON TOPOLOGICAL T_0-GROUPS

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There are two classes of continuous almost periodic functions on a topological T_0-group G, which are of special interest if we are concerned with connectedness properties of compactifications of G. One of these classes consists of those almost periodic functions on G, which may be regarded as the restrictions of continuous functions on a compactification of G, which in itself is a totally disconnected compact group. Regarding compactifications of G, which are connected compact groups, the other class of almost periodic functions is characterized in an analogous way.

First we define the notion of an elementary τ-almost periodic function. A continuous function on the topological group G is called *elementary τ-almost periodic* provided one of the following equivalent conditions holds

(1) *$\{f_a;\ a \in G\}$ is a finite set;*

(2) *$\{{}_af;\ a \in G\}$ is a finite set;*

(3) *$\{D_af;\ a \in G\}$ is a finite set;*

(4) *in G there exists an invariant subgroup H of finite index, such that f is constant on any coset of H;*

(5) *f is almost periodic, and $\alpha_f G$ is a finite group;*

(6) *f is almost periodic, and the Banach algebra $\mathfrak{A}_f$ is a finite direct sum of one-dimensional closed subalgebras;*

(7) *f is almost periodic and its set of values is finite.*

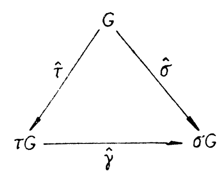
Concerning condition (5) $\alpha_f G$ denotes the compactification of G defined by the almost periodic function f. For example $\alpha_f G$ may be described as the completion of G/H in the invariant metric defined by f, if H denotes the maximal invariant subgroup such that f is constant on any coset.

Let $\mathfrak{A}$ be the commutative C^*-algebra of continuous almost periodic functions on the group with respect to the uniform norm. The Banach algebra $\mathfrak{A}_f$ of condition (6) is the closed subalgebra generated by f and invariant with respect to involution and right or left translations.

By $\mathfrak{A}^{(\tau)}$ we denote the closed subalgebra of $\mathfrak{A}$, which is generated by the elementary τ-almost periodic functions. The Banach algebra $\mathfrak{A}^{(\tau)}$ is a commutative C^*-algebra and invariant under right and left translations. The maximal ideal space of $\mathfrak{A}^{(\tau)}$ is a totally disconnected topological space in which the underlying topological space of G may be embedded by a continuous mapping $\hat{\tau}$. The image of $\hat{\tau}$ is everywhere dense in the maximal ideal space and the group operations translated by $\hat{\tau}$ from G to Im $\hat{\tau}$ are uniformly continuous in the usual weak topology for maximal ideals. Extending the group operations to the whole space by continuity we get a totally disconnected compact group τG, which is a compactification of the topological group G.

Theorem 1. *The compactification τG is universal for all totally disconnected compactifications of G.*

Let σG be a totally disconnected compact group, which is a compactification of G, and let $\hat{\sigma}$ denote the continuous embedding homomorphic mapping of G into σG. Theorem 1 states the existence of one and only one continuous homo-



morphism $\hat{\gamma}$ of τG into σG such that the diagram is commutative. The mapping $\hat{\gamma}$ is easily seen to be a mapping onto.

The compactification τG is called *the universal totally disconnected compactification of G*. The group τG is uniquely determined up to a topological isomorphism.

The elements of the C^*-algebra $\mathfrak{A}^{(\tau)}$ are called *τ-almost periodic functions*. Obviously there is a one-to-one correspondence between the τ-almost periodic functions on a topological group G and the continuous functions on its universal totally disconnected compactification τG.

Proposition 1. *A continuous function f on G is τ-almost periodic if and only if there exists a totally disconnected compactification σG and a continuous function g on σG such that $f(x) = g(\hat{\sigma}x)$, $x \in G$.*

A topological group G is called *minimally τ-almost periodic* provided $\mathfrak{A}^{(\tau)} = \{1\}$, and *maximally τ-almost periodic* if the continuous homomorphic mapping $\hat{\tau}$ is an algebraic isomorphism.

The following two propositions characterize minimally τ-almost periodic groups:

Proposition 2. *A group G is minimally τ-almost periodic if and only if there are no closed (open) invariant subgroups of finite index in G.*

Proposition 3. *A group G is minimally τ-almost periodic if and only if the C^*-algebra $\mathfrak{A}$ is direct indecomposable.*

Let βG denote the universal (sometimes called Bohr-) compactification of G.

The underlying compact topological space of βG is homeomorphic to the space of maximal ideals of the commutative C^*-algebra $\mathfrak{A}$. From Proposition 3 we get as a

Corollary. *The universal compactification βG is a compact connected group if and only if G is minimally τ-almost periodic.*

The following proposition states a necessary and sufficient condition for a topological group to be maximally τ-almost periodic:

Proposition 4. *A group G is maximally τ-almost periodic if and only if the intersection of all closed (open) invariant subgroups of finite index consists of the neutral element only.*

With regard to the second class of almost periodic functions we shall discuss in this note, we define the notion of elementary ζ-almost periodicity. A continuous function f on a topological group G is called *elementary ζ-almost periodic* provided one of the following equivalent conditions holds

(1) $\overline{\{f_a;\ a \in G\}}$ *is a compact connected space with respect to the uniform topology;*

(2) $\overline{\{{}_af;\ a \in G\}}$ *is a compact connected space with respect to the uniform topology;*

(3) $\overline{\{D_af;\ a \in G\}}$ *is a compact connected space with respect to the uniform topology;*

(4) *if H denotes the maximal invariant subgroup, such that f is constant on any coset of H, then there is no closed (open) invariant subgroup of finite index in G/H;*

(5) *f is almost periodic and $\alpha_f G$ is a connected group;*

(6) *f is almost periodic and the Banach algebra $\mathfrak{A}_f$ is direct indecomposable;*

(7) *there exists a connected compact group ηG, which is a compactification of G with the continuous embedding homomorphism $\hat{\eta}$, and a continuous function g on ηG, such that $f(x) = g(\hat{\eta}x)$, $x \in G$.*

By $\mathfrak{A}^{(\zeta)}$ we denote the closed subalgebra of $\mathfrak{A}$, which is generated by the elementary ζ-almost periodic functions. The Banach algebra $\mathfrak{A}^{(\zeta)}$ is a commutative C^*-algebra, which is invariant with respect to right and left translations. The elements of $\mathfrak{A}^{(\zeta)}$ we call *ζ-almost periodic functions.*

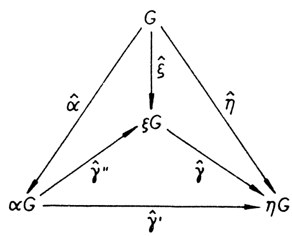
Remark. Note that ζ-almost periodic functions, which are not elementary ζ-almost periodic generally exist. By condition (7) it is impossible to find a connected compactification of G, such that these functions may be regarded as the restrictions of continuous functions on a connected compact group. There is a full analogy between τ-almost periodic functions and ζ-almost periodic functions on a topological group only in the case when each ζ-almost periodic function is elementary ζ-almost periodic.

Defining a group structure in the maximal ideal space of the commutative C^*-algebra $\mathfrak{A}^{(\zeta)}$ as indicated above we get a compactification ζG of G. The embedding continuous homomorphism of G in ζG is denoted by $\hat{\zeta}$.

Generally the compact group ζG is not connected.

Theorem 2. *The compactification ζG is universal for all connected compactifications of G, and it is the smallest one with this property.*

Let ηG be a connected compact group, which is a compactification of G, $\hat{\eta}$ denotes the continuous embedding homomorphism of G into ηG. Theorem 2 states the existence and uniqueness of a continuous homomorphism $\hat{\gamma}$ of ζG into ηG, which is easily verified to be an epimorphism, such that $\hat{\gamma} \circ \hat{\zeta} = \hat{\eta}$. Let moreover αG be a compactification of G which is universal for all connected compactifications ηG in the sense indicated above. If $\hat{\gamma}'$ denotes the unique continuous epimorphism of αG onto ηG with the property $\hat{\gamma}' \circ \hat{\alpha} = \hat{\eta}$, then Theorem 2 states the existence of one and only one continuous epimorphism $\hat{\gamma}''$ of αG onto ζG such that $\hat{\gamma} \circ \hat{\gamma}'' = \hat{\gamma}'$.



In the following two theorems we answer the question under what condition ζG is a connected compact group. In this case we call ζG, which is unique up to a topological isomorphism, the *universal connected compactification of G.*

Theorem 3. *The compact topological group ζG is connected if and only if $\mathfrak{A}^{(\zeta)}$ consists of elementary ζ-almost periodic functions.*

Theorem 4. *The compact topological group ζG is connected if and only if the commutative C^*-algebras $\mathfrak{A}^{(\tau)}$ and $\mathfrak{A}^{(\zeta)}$ are independent*: $\mathfrak{A}^{(\tau)} \cap \mathfrak{A}^{(\zeta)} = \{1\}$.

If G is a minimally τ-almost periodic group it follows from the definition of elementary ζ-almost periodicity that $\mathfrak{A}^{(\zeta)} = \mathfrak{A}$ and $\mathfrak{A}$ consists of elementary ζ-almost periodic functions. The corollary of Proposition 3 once more follows in the form $\beta G \underset{\text{top}}{\cong} \zeta G$ from Theorem 3.

Provided the elementary ζ-almost periodic functions and the elementary τ-almost periodic functions generate the algebra $\mathfrak{A} : \mathfrak{A} = \mathfrak{A}^{(\tau)} + \mathfrak{A}^{(\zeta)}$; the universal compactification βG is topologically isomorphic to a subdirect product of the groups τG and ζG. The converse is also valid.

Theorem 5. *The compact topological group βG is the topological direct product of the compact totally disconnected group τG and the compact group ζG if and only if the τ-almost periodic functions and the ζ-almost periodic functions generate the algebra $\mathfrak{A}$ of almost periodic functions: $\mathfrak{A} = \mathfrak{A}^{(\tau)} + \mathfrak{A}^{(\zeta)}$; and the subalgebras $\mathfrak{A}^{(\tau)}$ and $\mathfrak{A}^{(\zeta)}$ are independent: $\mathfrak{A}^{(\tau)} \cap \mathfrak{A}^{(\zeta)} = \{1\}$. The group ζG is connected in this case.*

If G denotes an abelian topological group, the equality $\mathfrak{A} = \mathfrak{A}^{(\tau)} + \mathfrak{A}^{(\zeta)}$ is always valid and from Theorems 4 and 5 follows the

Corollary. *The universal compactification βG of an abelian group G is the direct product of a totally disconnected and a connected compact group, if and only if G possesses a universal connected compactification.*

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