ON THE HAHN-MAZURKIEWICZ PROBLEM IN NON-METRIC SPACES

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Around 1914 H. Hahn [6] and S. Mazurkiewicz [22] independently characterized Peano continua, i.e. (Hausdorff) spaces X which are obtainable as images of the real line segment $I = [0, 1]$ under (continuous) maps $f : I \to X$ onto X. They proved that Peano continua coincide with metrizable locally connected continua.

It is natural to ask for a non-metric analogue of this theorem. However, such a result has not yet been obtained. It is to this open problem that we refer as to the Hahn-Mazurkiewicz problem. Recently the interest in this and related problems has been revived and it is the aim of this note to report on the progress made.

As the non-metric analogue of the arc I we consider here a (totally or linearly) ordered continuum C, also called a generalized arc. More generally, an ordered compactum K is a compact space whose topology can be derived from a total order $<$ as the induced order topology. The sub-basis for K is given by all sets of the form

$$(a, .) = \{t \mid t \in K,\ a < t\} \quad \text{and}$$
$$(., b) = \{t \mid t \in K,\ t < b\}, \quad a, b \in K .$$

Ordered continua C are connected ordered compacta and can be characterized as continua having at most two non-cut points (see e.g. [7]) or as locally connected continua irreducible between two points (see e.g. [33]). According to another characterization ordered continua are chainable locally connected continua [14].

There are many different ordered continua C, even many different ordered continua C with the property that all non-degenerate subcontinua of C are homeomorphic with C (see [1], [2], [25]). The arc I is the only ordered continuum which is metrizable. Notice also that ordered compacta K can be considered as closed subsets of ordered continua C.

The non-metric analogues of Peano continua are (Hausdorff) spaces X obtainable as images of ordered continua C under maps $f : C \to X$ onto $X, f(C) = X$. For these spaces we use here the abbreviation IOC (images of ordered continua). For spaces X obtainable as images of ordered compacta K under maps $f : K \to X$ onto $X, f(K) = X$, we use here the abbreviation IOK. Clearly, IOK's form a subclass of the class of (Hausdorff) compacta, while the IOC's form a subclass of the class of locally connected continua. In both cases we actually have proper subclasses.

The IOK's play a major role in the study of the IOC's. In 1960 P. Papić and the author [17] conjectured that a connected and locally connected IOK is in fact an IOC. This conjecture is still unsettled.

1. Irreducible and light maps

A map $f : Y \to X$ onto X is said to be *irreducible* provided no proper closed subset Y_0 of Y maps onto X under the restriction $f \mid Y_0$. Notice that irreducible maps map non-empty disjoint open sets into sets with disjoint non-empty interiors [4]. Furthermore, if a set $D \subset K$ has the property that $f(D)$ is dense in X and $f : K \to X$ is an irreducible map onto X, then D is dense in K [26].

A map $f : K \to X$ onto X is said to be *light* (in the sense of the order $<$ on K) provided there is no pair of (distinct) points $t_1, t_2 \in K$, $t_1 < t_2$, such that f maps the segment $[t_1, t_2] = \{t \mid t \in K,\ t_1 \leqq t \leqq t_2\}$ into a single point.

One can assume without loss of generality that every IOK X is the image of an ordered compactum K under a map $f : K \to X$ which is simultaneously light and irreducible.

2. Numerical invariants and mappings of ordered compacta

With every compactum Y we can associate several cardinals which are topologically invariant. The weight $w(Y)$ is the least cardinal of a basis for the topology of Y. The local weight $lw(Y)$ is the least upper bound of $w(Y, y)$, $y \in Y$, where $w(Y, y)$ is the character of Y at y, i.e. the least cardinal of a basis at the point $y \in Y$. The density or degree of separability $s(Y)$ is the least cardinal of a subset dense in Y. Finally, the Kurepa-Suslin number or degree of cellularity $c(Y)$ [10] is the least upper bound of cardinals of families of non-empty disjoint open subsets of Y.

Theorem 1. *If K is an ordered compactum and $f : K \to X$ is an irreducible mapping, then $s(K) = s(X)$ and $c(K) = c(X)$* (see [19], [26] and [4]).

Theorem 2 [19]. *If K is an ordered compactum and $f : K \to X$ is a light mapping onto an infinite space X, then $w(K) = w(X)$ and $lw(K) \leqq lw(X)$.*

Notice that a light irreducible map can actually increase the local weight (see [19]).

An immediate consequence of Theorems 1 and 2 is this

Theorem 3 [19]. *If X is an* IOK, *then X is the image of an ordered compactum K such that $w(K) = w(X)$, $lw(K) \leqq lw(X)$, $s(K) = s(X)$, $c(K) = c(X)$.*

Since w, lw, s and c are monotonously increasing functions on closed subsets of ordered compacta, one obtains immediately:

Theorem 4 [19]. *If X is an* IOK *and X_0 is a closed subset of X, then $w(X_0) \leqq w(X)$, $lw(X_0) \leqq lw(X)$, $c(X_0) \leqq c(X)$, $s(X_0) \leqq s(X)$.*

The first two of these inequalities are elementary and valid for arbitrary compacta. Not so the last two. E.g. if $\tau > \aleph_0$, then the cube I^τ has the the Suslin number $c(I^\tau) = \aleph_0$ [24], but I^τ contains closed subsets X_0 with $c(X_0) > \aleph_0$, e.g. the "long line" $[0, \omega_1]$. Thus, Theorem 4 implies that the cube I^τ is an IOK if and only if $\tau \leqq \aleph_0$.

3. Comparison of numerical invariants for IOK's

Using Theorem 3 P. Papić and the author ([18] and [19]) have proved this:

Theorem 5. *If X is an* IOK *then $c(X) \leqq \aleph_\alpha$, $\alpha \geqq 0$, if and only if each open subset $V \subset X$ is the union of $\leqq \aleph_\alpha$ closed subsets of X.*

In particular, note this:

Corollary 1 ([18] and [19]). *If X is an* IOK, *then X has the Suslin property (i.e. each family of disjoint open sets is at most countable) if and only if each open subset $V \subset X$ is an F_σ-set.*

An immediate consequence is

Theorem 6 [19]. *If X is an* IOK, *then*

$$lw(X) \leqq c(X) \leqq s(X) \leqq w(X).$$

Corollary 2 ([18] and [19]). *If X is an* IOK *with the Suslin property (in particular if X is separable), then X satisfies the first axiom of countability, i.e. $lw(X) \leqq \aleph_0$,*

Since every diadic compactum X has the Suslin property [24] and since diadic compacta satisfying the first axiom of countability are metrizable [5] (see also [4]), we see that Corollary 2 implies this:

Theorem 7 ([18] and [19]). *If a compactum X is at the same time diadic and an* IOK, *then X is metrizable.*

This theorem establishes a conjecture of P. S. Aleksandrov.

Another application of Theorem 3 yields this:

Theorem 8 [19]. *The following two statements are equivalent:*

(*i*) *If X is an* IOK, *then $c(X) = s(X)$.*

(*ii*) *If C is an ordered continuum, then $c(C) = w(C)$.*

(*ii*) is the generalized Suslin problem [10].

We now introduce two more invariants $\sigma^c(X)$ and $\sigma^o(X)$. We say that a family $\mathfrak{S}$ of subsets of X is a separating family provided for every pair of closed disjoint subsets $M, N \subset X$ there is a member $S \in \mathfrak{S}$ such that $X \setminus S$ is the union of two sets U, V with the property that $M \subset U$, $N \subset V$, $\bar{U} \cap V = U \cap \bar{V} = \emptyset$. The number $\sigma^o(X)$ $(\sigma^c(X))$ is defined as the least cardinal of a separating family $\mathfrak{S}$ consisting of open (closed) subsets.

Theorem 9 (see [15]). *If X is an* IOK, *then*

$$c(X) \leqq \sigma^o(X) \leqq \sigma^c(X) \leqq s(X) .$$

To show that $c(X) \leqq \sigma^o(X)$ consider a family of non-empty disjoint open subsets $\{U_\alpha\}$, $\alpha \in A$, and choose, for each $\alpha \in A$, a point $x_\alpha \in U_\alpha$. Since there is a separating family $\mathfrak{S}$ of cardinality $\leqq \sigma^o(X)$ consisting of open sets, we can separate $\{x_\alpha\}$ from $X \setminus U_\alpha$ by a member $S_\alpha \in \mathfrak{S}$. All S_α, $\alpha \in A$, are distinct because they belong to disjoint sets U_α. It follows that the cardinal $k(X) \leqq \sigma^o(X)$ and thus $c(X) \leqq \sigma^o(X)$. The same argument shows that $c(X) \leqq \sigma^c(X)$.

In order to show that $\sigma^o(X) \leqq \sigma^c(X)$, it is enough to apply Theorem 5 and the fact that $c(X) \leqq \sigma^c(X)$. It follows that each closed subset is the intersection of $\leqq \sigma^c(X)$ open subsets. Therefore, if we have a separating family $\mathfrak{F}$ of closed subsets and $k\mathfrak{F} = \sigma^c(X)$, then we obtain a separating family $\mathfrak{U}$ of the same cardinality as $\mathfrak{F}$ but consisting only of open sets (see [16]). This proves the inequality $\sigma^o(X) \leqq \sigma^c(X)$.

The inequality $\sigma^c(X) \leqq s(X)$ is proved in [15].

4. A metrization theorem for compacta

A new metrization theorem has been proved recently in [16]. To state this result we need a new numerical invariant $\mu(Y)$.

Given a compactum Y, consider all closed subsets $A \subset Y$ and for each A consider the space of components $Z(A)$. It is well-known that $Z(A)$ is always a zero-dimensional compactum. Let

$$\mu(Y) = \text{l.u.b.}\ \{w(Z(A))\} ,$$

where $A \subset Y$ runs through all closed subsets A of Y.

Theorem 10 (see [16]). *If Y is a (Hausdorff) compactum, then*

$$w(Y) \leqq \max\ \{\sigma^o(Y), \mu(Y)\} .$$

In particular, if $\sigma^o(Y) \leqq \aleph_0$ and $\mu(Y) \leqq \aleph_0$, then Y is metrizable.

Notice that always $c(Y) \leqq \mu(Y)$ [16]. Therefore, for X an IOK the hypothesis $c(X) = s(X)$ (equivalent to the generalized Suslin problem by Theorem 8) implies (by Theorem 9) that

$$\sigma^o(X) \leqq s(X) = c(X) \leqq \mu(X) .$$

so that $\max\ \{\sigma^o(X), \mu(X)\} = \mu(X)$. Generalizing [16] we can now prove:

Theorem 11. *The generalized Suslin problem is equivalent to the assertion that the inequality* $w(X) \leqq \mu(X)$ *holds for all* IOK's.

5. Frontiers of open sets and topological limits

In [15] the following result is proved.

Theorem 12. *Let* X *be an* IOK *and let* G *be an open subset of* X *which is the union of* $\leqq \aleph_\alpha$ *compact subsets. Then* $s(\mathrm{Fr}\ G) \leqq \aleph_\alpha$. *In particular, the frontier of an open* F_σ*-set in* X *is separable.*

A. J. WARD in [30] has studied extensively properties of limits of disjoint closed subsets in an IOK. Here is one of his results (stated under somewhat stronger assumptions).

Theorem 13 [30]. *Let* X *be an* IOK *and let* $\{F_\alpha\}$, $0 \leqq \alpha < \omega_\Theta$, *be a well-ordered sequence of disjoint non-empty closed subsets of* X. *If* $\Theta = 0$, *then* $\overline{\lim}$ top F_α *is separable. If* $\Theta \geqq 1$, *then* $\underline{\lim}$ top F_α *is a finite set.*

From the same paper we also quote:

Theorem 14 [30]. *Let* X *be an* IOK *and let* $\{F_\alpha\}$, $\alpha \in A$, *be a disjoint family of closed subsets with the property that*

$$F_{\alpha_0} \cap \mathrm{Cl}\left[\bigcup_{\alpha \neq \alpha_0} F_\alpha\right] = 0\,, \quad \textit{for each} \quad \alpha_0 \in A\,.$$

Then there is a family of disjoint open subsets V_α, $\alpha \in A$, *such that* $F_\alpha \subset V_\alpha$. *Furthermore, each* IOK *is a completely normal space.*

To prove the first statement (formulated by the author and proved by A. J. Ward) it is enough to prove it for ordered compacta, because the property is preserved under continuous maps of compacta.

As an easy application one obtains this:

Theorem 15 (see [30]). *Let* X *be an* IOK *and let* $\{F_\alpha\}$, $\alpha \in A$, *be a net of disjoint subcontinua* $F_\alpha \subset X$ *with the property that* $F_\alpha \cap \mathrm{Cl}\left[\bigcup_{\beta \neq \alpha} F_\beta\right] = 0$, *for each* $\alpha \in A$. *Then either* $\underline{\lim}$ top F_α *contains at most one point or for some* α_0 *the set* $\{\alpha \mid \alpha \geqq \alpha_0\}$ *is at most countable* (see [30]).

Indeed, if $x_1, x_2 \in \underline{\lim}$ top F_α, $x_1 \neq x_2$, then we can easily find disjoint neighborhoods U_1, U_2, $x_1 \in U_1$, $x_2 \in U_2$, which are open F_σ-sets. It follows, by Theorem 12, that Fr U_1 and Fr U_2 are separable spaces. Clearly, there is an $\alpha_0 \in A$ such that for each $\alpha \geqq \alpha_0$ the continuum F_α meets both U_1 and U_2 and therefore meets also Fr U_1. By assumption and Theorem 14 the sets $F_\alpha \cap \mathrm{Fr}\ U_1$, $\alpha \geqq \alpha_0$, can be surrounded by disjoint open sets of Fr U_1 and since Fr U_1 is separable, it follows that the set $\{\alpha \mid \alpha \geqq \alpha_0\}$ is at most countable.

6. The product theorem

Theorem 16. *If X and Y are infinite compacta and $X \times Y$ is an* IOK, *then both X and Y are metrizable.*

This theorem was first proved under stronger assumptions that $X \times Y$ is an IOC [17] (see also [12]). The present form was obtained independently by A. J. Ward [30] and L. B. Treybig [26]. For two other proofs of Theorem 16 see [15].

We also quote here a result of W. W. Babcock related to the product theorem:

Theorem 17 [3]. *Let m, n be integers such that $1 \leqq m < n$, let K_i, $i = 1, \ldots, m$, be separable ordered compacta and let X_j, $j = 1, \ldots, n$, be non-metrizable compacta. Then no open onto map $f: K_1 \times \ldots \times K_m \to X_1 \times \ldots \times X_n$ exists.*

This result suggests that one can expect interesting results by studying images of products of ordered compacta.

7. Connected IOK's

Treybig proved in [27] the following:

Theorem 18. *If a continuum X is an* IOK, *then $s(X) = w(X)$.*

The theorem was first proved by P. Papić and the author [19] under the additional assumption that X is locally connected (see also [17]).

Now Theorem 18 can be derived as a corollary of this more general

Theorem 19 (see [16]). *Let X be an* IOK *and let G be an open subset of X which is the union of $\leqq \aleph_\alpha$ compacta. If* Cl *G is connected, then $w(\mathrm{Fr}\ G) \leqq \aleph_\alpha$.*

Corollary 3 [16]. *If X is an* IOK *and G is an open F_σ-set with connected closure* Cl *G, then* Fr *G is metrizable.*

As a first step in the proof of Theorem 19 one applies Theorem 12 and concludes that $s(\mathrm{Fr}\ G) \leqq \aleph_\alpha$. It follows, by Theorem 9, that $\sigma^o(\mathrm{Fr}\ G) \leqq \aleph_\alpha$. Here is the product theorem an important tool. The arguments given in [16] readily extend to the present more general case. Finally, one applies Theorem 10 and concludes that $w(\mathrm{Fr}\ G) \leqq \aleph_\alpha$.

Using Theorem 18, Treybig proved in [27] this interesting

Theorem 20. *A connected* IOK *X is either metrizable or there exist two points $x_0, x_1 \in X$ (possibly identical) such that $X \setminus \{x_0, x_1\}$ is not connected.*

This theorem shows clearly the essentially new features of the non-metric case in the Hahn-Mazurkiewicz problem.

8. Locally connected IOK's and local peripheral metrizability

Using repeatedly normality we can easily show that every locally connected compactum admits a basis consisting of open connected F_σ-sets. This remark together with Theorem 19 yields this:

Theorem 21 [16]. *Every locally connected* IOK *X is locally peripherally metrizable, i.e. admits a basis of open sets with metrizable frontiers*[1]).

Corollary 4. *Every* IOC *is locally peripherally metrizable.*

9. Connectedness by generalized arcs

The classical arc theorem due to R. L. Moore [23] states that any two points of a Peano continuum are end-points of an arc I. In general we say that a space X is connected by ordered continua or connected by generalized arcs provided for any pair of points $x_0, x_1 \in X$ there is an ordered continuum $C \subset X$ such that x_0 and x_1 are the two end-points of C. Around 1941 A. D. Wallace raised the question whether every locally connected continuum is connected by generalized arcs. A counter-example was described in [13]. Notice that A. J. Ward proved [28] that all IOC's are connected by generalized arcs. It seems likely that all connected locally connected IOK's have the same property. An interesting theorem on the existence of generalized arcs in partially ordered continua was obtained by R. J. Koch in [8] (see also [31]).

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