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In: (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the second Prague topological symposium, 1966. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1967. pp. 122--128.

Persistent URL: http://dml.cz/dmlcz/700815

ON SEPARATION AND APPROXIMATION OF REAL FUNCTIONS DEFINED ON A CHOQUET SIMPLEX

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1. Introduction

Two important theorems concerning the space $C(X)$ of all real-valued continuous functions on a compact Hausdorff space X are (i) the Weierstrass-Stone theorem about linear sublattices of $C(X)$, (ii) the separation theorem of Katětov that, whenever $-f$, g are lower semicontinuous functions from X into $(-\infty, \infty]$ such that $f \leqq g$, one can find a function $h \in C(X)$ such that $f \leqq h \leqq g$. The main object of the present paper is to describe generalizations of these two theorems to the space of affine continuous functions on a Choquet simplex. In the case of Katětov's theorem we do slightly more than this, and describe *en passant* generalizations of Mokobodzki's two separation theorems [13] for convex functions.

A fuller account, with proofs, of the new separation theorems can be found in [7] and [8]; the same methods have since been shown, by Boboc and Cornea [4], to be applicable in a still more general context, important for potential theory. The extension to Choquet simplexes of the Weierstrass-Stone theorem is due to G. Vincent-Smith and the author [10].

2. Preliminaries

Let X be a compact Hausdorff space and let $C(X)$ be the Banach space of all real continuous functions on X. We shall denote by $M(X)$, $M_+(X)$, and $P(X)$ respectively the Radon, the positive Radon, and the probability Radon measures on X. If $f : X \to \to (-\infty, \infty]$ is a Borel measurable function bounded below and $\mu \in M_+(X)$, we shall denote by $\mu(f)$ the extended real number $\int^* f \,d\mu$; $\mu(-f)$ will then mean $-\mu(f)$. We recall that $M(X)$ is the Banach dual of $C(X)$ for the pairing $(\mu, h) \to \mu(h)$ and that $P(X)$ is a vaguely (i.e. weak*) compact subset of $M(X)$.

We consider a wedge $\mathscr{W}$ in $C(X)$ that contains the constant functions. For simplicity's sake we also suppose that $\mathscr{W}$ separates the points of X. To each point $x \in X$ we assign the set of measures

$$R_x \equiv R_x(\mathscr{W}) = \{\mu \in M_+(X) : \mu(f) \leqq f(x), \forall f \in \mathscr{W}\} .$$

By a $\mathscr{W}$-concave function we shall mean any semibounded Borel measurable extended-real-valued function f on X such that $\mu(f) \leqq f(x)$ whenever $x \in X$ and $\mu \in R_x$. $\mathscr{W}$-convex functions are defined analogously. We shall always assume that $\mathscr{W}$ is minimum-stable (min-stable) in the sense that

$$\min(f, g) \in \mathscr{W} \quad \text{whenever} \quad f, g \in \mathscr{W}\,.$$

Our first three objectives will be to characterize the continuous, the lower semicontinuous, and the upper semicontinuous $\mathscr{W}$-concave functions.

The following construction will be used. For each upper semicontinuous function $f : X \to [-\infty, \infty)$ and point $x \in X$ write

$$\hat{f}(x) = \inf \{g(x) : g \in \mathscr{W}, f \leqq g\}\,,$$

so that $\hat{f} : X \to [-\infty, \infty)$ is upper semicontinuous and

$$f(x) \leqq \hat{f}(x) \leqq \max_{y \varepsilon X} f(y)\,.$$

For fixed x the restriction to $C(X)$ of the map $f \to \hat{f}(x)$ is real-valued and sublinear By a standard Hahn-Banach argument one now has the following result.

Proposition 1. *For each function $f \in C(X)$ and point $x \in X$,*

$$\hat{f}(x) = \max \{\mu(f) : \mu \in R_x\}\,.$$

One can now deduce immediately two characterizations of the $\mathscr{W}$-concave continuous functions:

Corollary. *For each $f \in C(X)$ the following assertions are equivalent:*

(i) $f \in \overline{\mathscr{W}}$;
(ii) f *is* $\mathscr{W}$-*concave*;
(iii) $f = \hat{f}$.

This is in fact a trivial extension of *Satz* 7 of [2].

By a $\mathscr{W}$-affine function will be meant one that is $\mathscr{W}$-concave and also $\mathscr{W}$-convex. The $\mathscr{W}$-affine continuous functions are evidently just those in $\mathscr{A} \equiv \overline{\mathscr{W}} \cap (-\overline{\mathscr{W}})$.

A function defined merely on a non-empty closed subset E of X is called, by a convenient abuse of language, $\mathscr{W}$-concave ($\mathscr{W}$-convex etc.) if it is $\mathscr{W}_E$-concave ($\mathscr{W}_E$-convex etc.) with respect to the set of restrictions

$$\mathscr{W}_E \equiv \{f \mid E : f \in \mathscr{W}\}\,.$$

Thus to say that a function g on E is $\mathscr{W}$-concave means that g is a semibounded extended real-valued Borel measurable function such that $\mu(g) \leqq g(x)$ whenever $x \in E$ and $\mu \in R_x(\mathscr{W})$ with supp μ (the support of μ) a subset of E (so that $\mu(g)$ has a clear meaning).

A $\mathscr{W}$-stable set is, by definition, a non-empty closed subset E of X such that for each $x \in E$ and $\mu \in R_x(\mathscr{W})$ we have $\operatorname{supp} \mu \subseteq E$. The following construction is useful. Suppose that E is a $\mathscr{W}$-stable set and that

$$f : X \to (-\infty, \infty], \quad g : E \to (-\infty, \infty]$$

are lower semicontinuous and $\mathscr{W}$-concave, and that $g \leqq f \mid E$. Define $f_1 : X \to \to (-\infty, \infty]$ by

$$f_1(x) = \begin{cases} g(x) & (x \in E), \\ f(x) & (x \in X \setminus E). \end{cases}$$

Then f_1 is lower semicontinuous and $\mathscr{W}$-concave.

Finally, we recall that the Choquet boundary $\partial_{\mathscr{W}} X$ of X relative to $\mathscr{W}$ is defined as the set of all one-point $\mathscr{W}$-stable subsets of X (see [1, 2]).

3. Semicontinuous $\mathscr{W}$-concave functions

The following theorem extends and also sharpens a theorem of Mokobodski [13] concerning ordinary concave functions on a compact convex subset of a Hausdorff locally convex space.

Theorem 1. *Let $f : X \to (-\infty, \infty]$ be a lower semicontinuous $\mathscr{W}$-concave function and let $u \in C(X)$ be such that $u \leqq f$. Then there exists a $\mathscr{W}$-concave function $v \in C(X)$ such that $u \leqq v \leqq f$.*

The proof when $u < f$ is very simple. Proposition 1, with a little measure theory, implies here that $\hat{u}(x) < f(x)$ for all $x \in X$. The min-stability of $\mathscr{W}$ now implies, by a trivial covering argument, that there is a v in $\mathscr{W}$ such that $u < v < f$.

For the case $u \leqq f$ a well known approximation technique is used. Defining $u_0 = u - 1$ and $f_0 = f + 1$ one finds, by the preceding remarks, a $v_0 \in \mathscr{W}$ such that $u_0 < v_0 < f_0$. Proceeding inductively one obtains sequences $\{u_n\}$ etc., with $u_n \in C(X)$, f_n lower semicontinuous $\mathscr{W}$-concave, $v_n \in \overline{\mathscr{W}}$, and $u_n < v_n < f_n$, by the equations

$$u_{n+1} = \max\left(u - \frac{1}{2^{n+1}}, \quad v_n - \frac{1}{2^{n+1}}\right)$$

$$f_{n+1} = \min\left(f + \frac{1}{2^{n+1}}, \quad v_n + \frac{1}{2^{n+1}}\right)$$

together with the proof for the case $u < f$. One now has

$$u - \frac{1}{2^{n+1}} < v_{n+1} < f + \frac{1}{2^{n+1}}$$

and

$$\|v_{n+1} - v_n\| < \frac{1}{2^{n+1}}.$$

Consequently $v \equiv \lim_{n\to\infty} v_n$ exists and has the desired properties.

We have immediately the

Corollary 1. *Let $f : X \to (-\infty, \infty]$ be a lower semicontinuous function. Then f is $\mathscr{W}$-concave if and only if f is the pointwise limit of an increasing filtering family of elements of $\mathscr{W}$.*

For the case of ordinary concave functions on a compact convex subset of a Hausdorff locally convex space this corollary is a result of Mokobodzki [13].

Corollary 2. *Let E be a $\mathscr{W}$-stable subset of X and let $K \neq \emptyset$ be a compact subset of X disjoint from E. Then there exists a $v \in \overline{\mathscr{W}}$ such that $0 \leqq v \leqq 1$, $v(x) = 0$ for all $x \in E$, and $v(x) = 1$ for all $x \in K$. If E is also a G_δ then we can choose $w \in \overline{\mathscr{W}}$ such that $0 \leqq w \leqq 1$, $w(x) = 0$ for all $x \in E$, and $w(x) > 0$ for all $x \in X \setminus E$.*

This result appears to yield new information even in the classical Krein-Milman context. In some respects Corollary 2 can be sharpened in special cases: (a) sup-norm algebras (not dealt with here), (b) Choquet simplexes, and spaces satisfying the condition (S) (see below).

Closely related to Theorem 1 is the following mild generalization of a result of Choquet (see appendix B 14 of [6]).

Theorem 2. *In the relative topology the Choquet boundary $\partial_{\mathscr{W}}X$ of X is a Baire space.*

For the proof, see [9].

Much less useful than Theorem 1 is

Theorem 3. *A function $f : X \to [-\infty, \infty)$ is upper semicontinuous and $\mathscr{W}$-concave if and only if it is the pointwise infimum of a non-empty family of elements of $\mathscr{W}$.*

Like Theorem 1, this was suggested by a result of Mokobodzki [13].

4. A separation property

In this section we suppose that $\mathscr{W}$ satisfies the separation condition

(S): *whenever $-f, g \in \mathscr{W}$ with $f < g$ we can find a $\mathscr{W}$-affine continuous function h such that $f < h < g$.*

It is easy to show that this condition is realized for the wedge of all continuous concave functions on a Choquet simplex (see [7, 8]). It is also realized by certain wedges of superharmonic functions (see [3, 4, 8]).

The approximation technique used above to prove Theorem 1, suitably applied to the present context, yields

Theorem 4. *Suppose that $\mathscr{W}$ has property* (S) *and that $-f, g : X \to (-\infty, \infty]$ are $\mathscr{W}$-concave lower semicontinuous functions such that $f \leqq g$. Then there exists a function $h \in \mathscr{A}$ such that $f \leqq h \leqq g$.*

Corollary 1. *Let $\mathscr{W}, f, g$ be as in theorem* 4. *Let E be a $\mathscr{W}$-stable subset of X and let $h : E \to R$ be $\mathscr{W}$-affine, continuous and such that*

$$f \mid E \leqq h \leqq g \mid E .$$

Then there is a function $\bar{h}$ in $\mathscr{A}$ that extends h and satisfies

$$f \leqq \bar{h} \leqq g .$$

This corollary has many applications. In the definitive paper [11] of Effros on the facial structure of simplexes a special case of this corollary (Effros' theorem 2.4) plays a decisive part.

Theorem 4 was first proved in [7] (but compare [3]) for the special case of the ordinary concave functions on a Choquet simplex; in that situation the conclusion of Theorem 4 was shown there to characterize Choquet simplexes among the compact convex sets.

The following result is a special case of a theorem of E. B. Davies [5].

Proposition 2. *Let Q be a (closed) face of a Choquet simplex X and let $K \neq \emptyset$ be a compact subset of X disjoint from Q. Then there is a nonnegative continuous real affine function h on X that vanishes identically on Q and is >0 on K. If Q is also a G_δ set then there is a continuous affine function h on X that vanishes on Q and is >0 on $X \setminus Q$.*

A different proof, based on the work of Effros, was discovered independently by Lazar.

5. A Weierstrass-Stone theorem for simplexes

The result to be described here is a joint work with G. Vincent-Smith; a fuller account, with proofs, will appear in [10].

We consider a Choquet simplex X, and denote by X_e the set of all extreme points of X. By $\mathscr{A}(X)$ we understand the linear space of all real continuous affine functions on X. We consider a linear subspace L of $\mathscr{A}(X)$ that has the Riesz decomposition property: i.e. whenever $u_1, u_2, v_1, v_2 \in L$ with

$$\max(u_1, u_2) \leqq \min(v_1, v_2)$$

we can find a function $w \in L$ such that

$$\max(u_1, u_2) \leqq w \leqq \min(v_1, v_2).$$

It is a result of Lindenstrauss [12] that $\mathscr{A}(X)$ itself has this property (see [7] for a simpler proof). It is easy to see that if L has the Riesz property and contains the constant functions, then the closure of L in $C(X)$ has the property. Accordingly we take L to be already closed. By a result of Riesz the Banach dual L^* of L is a vector lattice whose positive cone has as base the set

$$Y = \{F \in L^* : F \geqq 0, \ \|F\| = 1\}.$$

This set Y is convex and compact for the topology $\sigma(L^*, L)$, and is in fact a Choquet simplex. The pairing between L and L^* induces an identification of L with $\mathscr{A}(Y)$. The injection $L \to \mathscr{A}(X)$ has a dual $\mathscr{A}(X)^* \to L^*$ which induces a $\sigma(L^*, L)$-continuous map $\pi : X \to Y$ such that $\pi(X) = Y$.

To simplify the discussion we suppose now that L separates the points of X_e; the general case will be discussed in [10].

Proposition 3. *The following properties are equivalent*:

(i) $\pi(X_e) \subseteq Y_e$;
(ii) *if* $u \in X_e$, $v \in X$, $u \neq v$ *then* $\pi(u) \neq \pi(v)$ (*or, equivalently, for some* $g \in L$ *we have* $g(u) \neq g(v)$);
(iii) *if* $x \in X_e$ *and* $f \in L$ *with* $f(x) = 0$ *then there is a* $g \in L$ *such that* $g \geqq \max(f, 0)$ *and* $g(x) = 0$.

This proposition is proved by a simple discussion of extreme points together with an application of Theorem 4. Property (i) is sometimes stated by saying that π is pure-state-preserving.

Theorem 5. ("Weierstrass-Stone"). *Suppose that X is a Choquet simplex and that L is a closed linear subspace of $\mathscr{A}(X)$ that has the Riesz decomposition property, contains the constant functions, separates the points of X_e and satisfies the conditions of Proposition* 3. *Then $L = \mathscr{A}(X)$.*

The conditions that L separates the points of X_e and contains the constant functions can both be relaxed, but these questions will not be considered here (see [10]).

The proof of Theorem 5 depends upon showing that, whenever $f \in \mathscr{A}(X)$, the set of functions

$$\{g \in L : g < f\}$$

is an increasing filtering family. This is proved by methods from Choquet boundary theory. Once this has been done the desired result follows by use of Dini's theorem and Bauer's minimum theorem.

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