Peter J. Nyikos

Strongly zero-dimensional spaces

In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 341--344.

Persistent URL: http://dml.cz/dmlcz/700810

STRONGLY ZERO-DIMENSIONAL SPACES

P. NYIKOS

Pittsburgh

In this paper I will review some old and not-so-old results on strongly zero-dimensional spaces, but my main purpose is to bring out two important unsolved problems and their many ramifications.

A strongly zero-dimensional space is a Tychonoff space whose Čech-Stone compactification βX is totally disconnected. There are many varied conditions which are equivalent to this one:

Theorem. *Let X be a Tychonoff space. The following conditions are equivalent*:

1. *βX is totally disconnected.*

2. *Given two disjoint zero-sets Z_1 and Z_2 of X, there exists a clopen (closed-and-open) subset C of X such that $Z_1 \subset C$, $Z_2 \cap C = \emptyset$.*

3. *Every zero-set of X is a countable intersection of clopen sets.*

4. *In the ring $C(X)$ of continuous real-valued functions, any two elements which generate the same principal ideal are associates. (One could also use the ring $C^*(X)$ of all bounded continuous real-valued functions here.)*

5. ad $C^*(X) = 0$. *In other words, every bounded continuous real-valued function on X is a uniform limit of continuous functions of finite range.*

There are many other equivalent conditions (cf. [5], [8]).

There is a kind of covering dimension which coincides with the analytic dimension of $C^*(X)$ and also with the Lebesgue covering dimension of βX [3, Chapter 16]. It is defined the same way as Lebesgue covering dimension except that cozero-sets are used in the place of open sets. A number of authors have taken to using "dim" for this kind of dimension on the grounds that it coincides with Lebesgue covering dimension for normal spaces and in some respects is more satisfactory than Lebesgue dimension for non-normal spaces. Under this system, dim $X = 0$ if, and only if, X is strongly zero-dimensional. In this paper, I will use the notation "dim" in this sense. I will also use the notation "D_S" for "strongly zero-dimensional".

One attractive feature of strong zero-dimensionality is the natural way it comes up, as illustrated by the following results. A space is D_S and metrizable if, and only if, it is T_0 and has a σ-locally finite base of clopen sets [4]. A space is D_S and paracompact if, and only if, it is T_1 and every open cover has a locally finite clopen refinement [1]. A space is D_S, normal, and countably paracompact if, and only if, it is Tychonoff and every countable open cover has a locally finite clopen refinement. In each case, we take either the definition of a class of spaces or a theorem which gives an equivalent condition (like the Nagata-Smirnov metrization theorem) and substitute the word "clopen" for "open" in the right place. (Sometimes, as above, we can even weaken the usual separation condition.) The same can be done for the definitions of perfect normality, collectionwise normality, and normality.

An even more attractive feature of strong zero-dimensionality is that conditions which seem much more special often turn out to be equivalent to it. For example, it is not hard to show that a space which admits a non-Archimedean metric[1]) is D_S; but who would suspect that the converse is also true (as shown by de Groot [4])? Similarly, on hearing the condition, "every open cover can be refined to a partition into clopen sets", one might think, what a pity this convenient property is so much more special than strong zero-dimensionality and paracompactness together! Yet the two latter conditions do imply the former.

But now let me get on to the two main unsolved problems. Briefly, they are:

(1) Is every product of D_S spaces itself D_S?

(2) Is every closed subspace of a realcompact D_S space itself D_S?

I became interested in the second problem through some work which led me last year to a negative solution to a long-standing problem: can every real-compact space with a base of clopen sets be embedded as a closed subspace of a product of countable discrete spaces? The problem arose from an article by Engelking and Mrówka [2] where spaces admitting such an embedding were called "N-compact" spaces, after the countable discrete space N of natural numbers.

At the time I started work on this problem, the following facts were known [5]: every D_S realcompact space is N-compact; every N-compact space has a base of clopen sets and is realcompact; and a metric space Δ described by Roy [9] is realcompact and has a base of clopen sets, but is not D_S : $\dim \Delta = 1$. In short, the class of N-compact spaces is sandwiched between two similar, but distinct, classes of spaces. I showed Δ is not N-compact, thus settling the nature of one containment but leaving unanswered the question of whether every N-compact space is D_S. That question remains open.

In fact, that question is equivalent to (2). To see this, one need only assemble the following information: every D_S realcompact space is N-compact; a closed subspace of an N-compact space is itself N-compact; and the product of arbitrarily many copies of the space N is D_S because every continuous real-valued function factors through a countable subproduct.

[1]) A metric d is *non-Archimedean* if it satisfies the strong triangle inequality: $d(x, z) \leqq \leqq \max \{d(x, y), d(y, z)\}$ for all x, y, z.

Since any product of N-compact spaces is N-compact, an affirmative solution to (2) would show also that any product of D_S realcompact spaces is itself D_S, a question which is itself open, even for finite products.

All results obtained thus far are trifling compared to what we would know if the answer to either (1) or (2) turned out to be yes. It is even unknown whether every metrizable N-compact space, or every separable normal N-compact space, is D_S. About the only general result on (2) is that every Lindelöf N-compact space is D_S.

K. Morita has shown that an arbitrary product of D_S Lindelöf Σ-spaces is D_S. Using the characterization of D_S paracompact spaces given above, and methods in Nagami's paper on Σ-spaces [7], one can show that a countable product of D_S paracompact Σ-spaces is itself a D_S paracompact Σ-space.

As for finite products, every general rule seems to be the zero-dimensional case of some theorem for arbitrary finite dimensions:

Theorem (Morita) [6]. *Let X be a paracompact space and let Y be a countable union of locally compact paracompact spaces. Then*

$$\dim(X \times Y) \leqq \dim X + \dim Y.$$

Theorem (Kodama). *Let X and Y be such that $X \times Y$ is countably paracompact and normal, with Y metrizable. Then*

$$\dim(X \times Y) \leqq \dim X + \dim Y.$$

Theorem (Morita). *Let X be an M-space and let Y be either metric or locally compact and paracompact. Then*

$$\dim(X \times Y) \leqq \dim X + \dim Y.$$

And, of course, there is the trivial result that if Y is discrete, then $\dim(X \times Y) = \dim X$ for any X at all. Otherwise the results consist of merely knocking down specific spaces. For example, P. Roy and I recently showed that the Sorgenfrey plane[2]) is a D_S space. But our proof does not even extend to Sorgenfrey 3-space, nor does a proof by R. Heath and D. Lutzer of the same fact[3]).

But unless one is looking for a counterexample, it seems almost a waste of time to work on finite products when there remains unanswered the sweeping question of whether every N-compact space is strongly zero-dimensional. There is a theorem by Herrlich that a Hausdorff space with a base of clopen sets is N-compact if, and only if, every clopen ultrafilter on X with the countable intersection property is fixed [5].

---

[2]) The Sorgenfrey line S is a D_S Lindelöf space, while the Sorgenfrey plane $S \times S$ is not even normal and does not come under any of the general results.

[3]) Added in proof: M. Mrówka has recently obtained the result that all powers of the Sorgenfrey line are strongly zero-dimensional.

Perhaps one could take any space with a base of clopen sets and, on the assumption that it is not D_S, exhibit a free clopen ultrafilter with the countable intersection property. That seems to be the most promising approach so far.

References

[1] *R. L. Ellis:* Extending continuous functions on zero-dimensional spaces. Math. Ann. *186* (1970), 114—122.

[2] *R. Engelking and S. Mrówka:* On *E*-compact spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. *6* (1958), 429—436.

[3] *L. Gillman and M. Jerison:* Rings of Continuous Functions. Van Nostrand Co., Princeton, New Jersey, 1960.

[4] *J. de Groot:* Non-Archimedean metrics in topology. Proc. Amer. Math. Soc. *7* (1956), 945—953.

[5] *H. Herrlich:* $\mathfrak{E}$-kompakte Räume. Math. Z. *96* (1967), 228—255.

[6] *K. Morita:* On the product of paracompact spaces. Proc. Japan Acad. *39* (1963), 559—563.

[7] *K. Nagami:* Σ-spaces. Fund. Math. *65* (1969), 169—192.

[8] *P. Nyikos:* Not every zero-dimensional realcompact space is *N*-compact. Bull. Amer. Math. Soc. *77* (1971), 392—396.

[9] *P. Roy:* Nonequality of dimensions for metric spaces. Trans. Amer. Math. Soc. *134* (1968), 117—132.