Jan M. Aarts

Complementary inductive invariants and dimension

In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 17--18.

Persistent URL: http://dml.cz/dmlcz/700798

COMPLEMENTARY INDUCTIVE INVARIANTS AND DIMENSION

J. M. AARTS

Delft

All spaces under discussion are assumed to be metrizable. Let $\mathscr{P}$ be a non-empty class of spaces which is closed for topological mappings. Then the following topological invariants can be defined.

(1) The strong (weak) inductive invariant $\mathscr{P}$-Ind X ($\mathscr{P}$-ind X) induced by the class $\mathscr{P}$ is inductively defined in a similar way as Ind X (ind X), but starting with the definition that $\mathscr{P}$-Ind X ($=\mathscr{P}$-ind X) $= -1$ iff $X \in \mathscr{P}$.

Of course, inductive dimension ($\mathscr{P} = \{\emptyset\}$) is the best explored inductive invariant. The concept of an inductive invariant has been introduced by Lelek [5].

(2) The deficiency of X with respect to $\mathscr{P}$ is defined as follows: $\mathscr{P}$-def $X \leqq n$ if there exists $Y \in \mathscr{P}$ such that $X \subset Y$ and $\dim Y \setminus X \leqq n$.

The case that $\mathscr{P}$ is the class of all compact spaces was first discussed by de Groot [2]. To these invariants we add

(3) The surplus of X with respect to $\mathscr{P}$ is defined by $\mathscr{P}$-sur $X \leqq n$ if there exists $Y \in \mathscr{P}$ such that $Y \subset X$ and $\dim X \setminus Y \leqq n$.

$\mathscr{P}$-def $X = n$, $\mathscr{P}$-def $X = \infty$ etc. are defined as usual. E.g. $\{\emptyset\}$-def $X = \infty$, whenever $X \neq \emptyset$.

It can be shown quite easily that $\mathscr{P}$-Ind $X \leqq \mathscr{P}$-sur X for every space X, if the class $\mathscr{P}$ is closed monotone (i.e. $Z \in \mathscr{P}$, whenever $Y \in \mathscr{P}$ and Z is a closed subset of Y). Furthermore, $\mathscr{P}$-Ind $X \leqq \mathscr{P}$-def X for every space X, if the class $\mathscr{P}$ is closed monotone and open monotone.

By $\mathscr{M}(\alpha)$ and $\mathscr{A}(\alpha)$ we denote the class of all sets of absolute multiplicative and additive Borel class α respectively. (See [4] for definitions. Recall that $\mathscr{A}(0) = \{\emptyset\}$, $\mathscr{M}(0)$ is the class of compact spaces, $\mathscr{A}(1)$ is the class of σ-locally compact spaces [7] and $\mathscr{M}(1)$ is the class of topologically complete spaces.)

Theorem 1. *Let* $\mathscr{P} = \mathscr{A}(\alpha)$ *or* $\mathscr{P} = \mathscr{M}(\alpha)$ *for* $\alpha \geqq 2$. *Then* $\mathscr{P}$-Ind $X \leqq n$ *if and only if there exist* $Y, Z \in \mathscr{P}$ *satisfying* $Y \subset X \subset Z$ *and* $\dim Z \setminus Y \leqq n$. *In particular* $\mathscr{P}$-Ind $X = \mathscr{P}$-def $X = \mathscr{P}$-sur X *for every space* X.

Theorem 2. (See [1].) $\mathscr{M}(1)$-Ind $X = \mathscr{M}(1)$-def X *for every space* X.

Theorem 3. $\mathscr{A}(1)$-Ind $X = \mathscr{A}(1)$-sur X *for every space* X.

Problems. Are the equalities $\mathscr{M}(1)$-Ind $X = \mathscr{M}(1)$-sur X and $\mathscr{A}(1)$-Ind $X = \mathscr{A}(1)$-def X valid for every space X ? To prove the second equality for separable spaces is a problem[1]) posed by Nagata [6]. As follows from the corollary below these equalities are closely related. It is a long unsolved problem whether or not $\mathscr{M}(0)$-ind $X = \mathscr{M}(0)$-def X for every separable space X ([2], [3]).

Definition. Let $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{R}$ be topologically closed classes of spaces. $\mathscr{P}$ and $\mathscr{Q}$ are *complementary with respect to* $\mathscr{R}$ if for every $Z \in \mathscr{R}$ and for all X and Y with $X \cup Y = Z$ and $X \cap Y = \emptyset$ the equality $\mathscr{P}$-Ind $X = \mathscr{Q}$-Ind Y holds.

Theorem 4. *$\mathscr{A}(1)$ and $\mathscr{M}(1)$ are complementary with respect to $\mathscr{M}(0)$. $\mathscr{A}(\alpha)$ and $\mathscr{M}(\alpha)$ are complementary with respect to $\mathscr{M}(1)$ for $\alpha \geqq 2$.*

Corollary. *If $\mathscr{M}(1)$-Ind $X = \mathscr{M}(1)$-sur X for every separable space X, then $\mathscr{A}(1)$-Ind $X = \mathscr{A}(1)$-def X for every separable space X.*

Example. It is known [1] that for the product X of the rationals and the n-dimensional cube I^n, we have $\mathscr{M}(1)$-Ind $X = n$.

By Theorem 4 it follows that for the product Y of the irrationals and I^n we have $\mathscr{A}(1)$-Ind $Y = n$.

The proofs of Theorems 1, 3, and 4 will be published in forthcoming papers.

References

[1] *J. M. Aarts:* Completeness degree. A generalization of dimension. Fund. Math. *63* (1968), 27–41.

[2] *J. de Groot:* Topologische studien. Thesis, Groningen, 1942.

[3] *J. de Groot and T. Nishiura:* Inductive compactness as a generalization of semi-compactness. Fund. Math. *58* (1966), 201–218.

[4] *C. Kuratowski:* Topologie I. Warszawa, 1958.

[5] *A. Lelek:* Dimension and mappings of spaces with finite deficiency. Colloq. Math. *12* (1964), 221–227.

[6] *J. Nagata:* Some aspects of extension theory in general topology. Contributions to Extension Theory of Topological Structures (Proc. Sympos., Berlin, 1967). Deutsch. Verlag Wissensch., Berlin, 1969, 157–161.

[7] *A. H. Stone:* Absolute F_σ spaces. Proc. Amer. Math. Soc. *13* (1962), 495–499.

[1]) Added in proof: This problem has been solved in the negative by J. M. Aarts and T. Nishiura.