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POINT-COUNTABLE BASES AND QUASI-DEVELOPMENTS

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1. Introduction

An important class of topological spaces is the one with point-countable bases. (We will refer to these as point-countable spaces).

In this paper we discuss some properties of these spaces and of some spaces with more restrictive base properties and relate these to spaces with corresponding type covering properties.

In a recent survey paper of the author [1], some of the important properties of point-countable spaces are listed. By using standard techniques for minimal properties of topological spaces as in [5], one can show that spaces minimal with respect to being Hausdorff and having a point-countable base are semiregular and feebly compact. It would be interesting to know if such spaces are necessarily H-closed. Corson and Michael [7] showed that countably compact point-countable spaces have a countable base. Modifying their techniques one can show that a topological space that is H-closed, Hausdorff, semiregular and has a base such that every point is in the closure of only a countable number of members of the base has a countable base. However there is an example of Miščenko [11] of a point countable Hausdorff, H-closed space that does not have a countable base. This space is not semiregular nor is each point in the closure of just a countable number of members of any base.

2. Quasi-developments

Recently E. E. Grace and H. R. Bennett [3] have introduced a generalization of the developable space.

Definition 1. A sequence $\mathscr{G}_1, \mathscr{G}_2, \ldots$ of collections of open sets of a topological space $(X, \mathscr{T})$ is called a *quasi-development* for $(X, \mathscr{T})$ provided that if $x \in T \in \mathscr{T}$, there exists n and G such that $x \in G \in \mathscr{G}_n$, and if $x \in G \in \mathscr{G}_n$ then $G \subset T$. We will refer to each $\mathscr{G}_k$ as a *collection of the quasi-development*.

We note that a quasi-developable space is a weak σ-space (follows from Lemma 3 below) [1]) and a point-countable weak σ-space is quasi-developable. From the latter result follows the result of Okuyama [13] that a collectionwise normal T_1 σ-space is metrizable iff it has a point-countable base and the result of Heath [9] that a T_3 stratifiable space is metrizable iff it has a point-countable base.

Definition 2. A topological space $(X, \mathcal{T})$ is a *weak σ-space* if it has a σ-disjoint network where each disjoint family is discrete with respect to some open subspace containing all members of the family.

As the developable space has the property of converting certain covering properties to corresponding base properties, the quasi-developable space has the property of converting certain hereditary covering properties to corresponding base properties. This will be clarified in Theorem 4.

Bennett and Lutzer [4] have showed the equivalence of a space being quasi-developable with having a θ-base. The concept of θ-base was introduced by Worrell and Wicke [14].

The following definitions will also be useful.

Definition 3. A family of sets will be said to be[2])

1. σ-0 if it is σ-disjoint.
2. σ-1 if it is σ-relatively locally finite, i.e., if the union of a denumerable number of families each locally finite in its union.
3. σ-2 if it is σ-point finite.
4. σ-3 if it is point-countable.

A topological space will be said to be *σ-k refinable* if every open cover has a σ-k open refinement.

Definition 4. A topological space is said to be *CN-k* if for every discrete family $\{D_a\}$ there is a σ-k family $\{G_a\}$ such that $D_a \subset G_a$ and $D_b \cap G_a = \emptyset$ for $a \neq b$. *HCN-k* will be used to indicate that every subspace is *CN-k*.

We note that a metacompact space is *CN*-2 and a point-countable space is *CN*-3. The first result is in Krajewski and Smith [9].

Theorem 1. *A countably paracompact (countably metacompact) σ-k refinable space is paracompact (metacompact) for $k = 0, 1$ (for $k = 2$).*

We note the result for $k = 0$ is due to Nagami [12].

Theorem 2. *A perfectly normal T_1 space with a σ-k base is metrizable for* $k = 0, 1$.

Proof. For $k = 0$ it is proved in [1] and the other case is similar.

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[1]) Result obtained independetly by E. S. Berney.

[2]) In subsequent statements $k = 0, 1, 2$ or 3 unless specifically indicated otherwise.

Theorem 3. *Let $(X, \mathcal{T})$ be quasi-developable and satisfy HCN-k, then $(X, \mathcal{T})$ has a σ-k base and is hereditary σ-k refinable.*

The proof is a modification of the proof of Theorem 2 in [2] in which the θ-base replaces the σ-point finite base. The following lemma is needed and its proof is similar to Lemma 1 in [2].

Lemma 3. *Let $\mathscr{V}$ be a family of open subsets of a topological space $(X, \mathcal{T})$. Then there exists a σ-disjoint family $\mathscr{M} = \bigcup_1^\infty \mathscr{M}_n$ such that*

(1) *Each $\mathscr{M}_n$ is discrete with respect to some open subspace G_n of X. (It is understood that if $M \in \mathscr{M}_n$ then $M \subset G_n$.)*

(2) *If x is contained in a finite number of members of $\mathscr{V}$ and if $x \in V \in \mathscr{V}$ then there exists $M \in \mathscr{M}$ such that $x \in M \subset V$.*

(3) *If x is not contained in at least one member of $\mathscr{V}$ or if x is contained in an infinite number of members of $\mathscr{V}$ then there does not exist $M \in \mathscr{M}$ such that $x \in M$.*

Corollary 3. *A CN-0 (Bing) or a CN-1 normal Moore space is metrizable.* See [6].

Theorem 4. *In a quasi-developable space hereditary σ-k refinability is equivalent to having a σ-k base.*

3. σ-locally countable bases[3])

To the author's knowledge not much is known about spaces with σ-locally countable basis. However Fedorčuk [8] proved that paracompact Hausdorff spaces with this base property are metrizable. Like developable spaces it seems to have the property of converting certain covering properties into corresponding base properties. For instance it can be proved that if such spaces are σ-*n* refinable (weakly θ-refinable [4]) they have a σ-*n* base (θ-base).

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3) The author is indebted to F. D. Tall for pointing out an error in this section.

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