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CARDINAL FUNCTIONS ON PRODUCTS

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Cardinal functions, i.e., functions defined on classes of topological spaces and having cardinal numbers as values, can be used to unify a diversity of cardinality problems arising in general topology (cf. [5]). Thus e.g., many problems concerning product spaces have the following general form:

Let us be given a cardinal function φ on a productive class $\mathscr{C}$, and spaces $R_i \in \mathscr{C}$, $i \in I$. Evaluate or estimate $\varphi(R)$, where $R = \bigtimes_{i \in I} R_i$, in terms of the values $\varphi(R_i)$ and $|I|$. (We always assume that none of the spaces R_i is indiscrete.) We shall mention several results of this kind in this lecture.

If φ is one of the functions w (weight), π (π-weight) or χ (character) defined on the class $\mathscr{T}$ of all topological spaces, or the function ψ (pseudo-character) defined on the class $\mathscr{T}_1$ of T_1 spaces, we have the following exact formula:

$$\varphi(R) = |I| \, . \sup \{\varphi(R_i) : i \in I\} \, .$$

A different exact formula can be given for the density function d on the class of spaces containing two disjoint non-empty open sets as follows:

$$\mathrm{d}(R) = \log |I| \, . \sup \{\mathrm{d}(R_i) : i \in I\} \, .$$

Here $\log \alpha = \min \{\beta : 2^\beta \geqq \alpha\}$ and the $\leqq$ holds on the whole $\mathscr{T}$, according to a well-known theorem of E. S. Pondiczery and E. Hewitt (see [9] or [4]).

The case of the cellularity number c is especially interesting, because it is closely connected to undecidable set theoretic problems, such as the Suslin hypothesis. Indeed, G. Kurepa [6] has shown that if this hypothesis fails, i.e., there exists a Suslin continuum X, then we have $\mathrm{c}(X) = \omega$ but $\mathrm{c}(X \times X) = \omega_1$. On the other hand Martin's axiom (see [8], [5], or [2]) implies that $\mathrm{c}(R) = \omega$ if $\mathrm{c}(R_i) = \omega$ for all $i \in I$. We do not know whether it is consistent to assume

$$\mathrm{c}(R) = \sup \{\mathrm{c}(R_i) : i \in I\}$$

on $\mathscr{T}$. However, the following estimate, due to G. Kurepa [7] (see also [3]) is valid on $\mathscr{T}$ without any special set-theoretic assumptions:

$$\mathrm{c}(R) \leqq \sup \{2^{\mathrm{c}(R_i)} : i \in I\} \, .$$

I do not know whether this formula can be sharpened as follows:

$$c(R) \leqq \sup \{c(R_i)^+ : i \in I\}$$

(here α^+ is the successor cardinal of α).

Concerning the spread function s, defined as the supremum of the cardinalities of discrete subspaces, the following formula has been recently established by A. Hajnal and the present author [1] for the class $\mathscr{T}_2$ of Hausdorff spaces:

$$s(R) \leqq |I| \,.\, \sup \{2^{s(R_i)} : i \in I\}\,.$$

This settles a conjecture from [5], Chapter 4. The Sorgenfrey line S is known to have $s(S) = \omega$ and $s(S \times S) = 2^\omega$, which shows that this estimate cannot be improved. The proof of this result is quite difficult, requiring the construction of a very complicated ramification system.

We conjecture that a similar formula is valid for the Lindelöf degree $\mathscr{L}$, too ($\mathscr{L}(X) = \min \{\alpha : X$ is α-Lindelöf$\}$), however we cannot even show that the product of two ω-Lindelöf spaces is 2^ω-Lindelöf.

References

[1] *A. Hajnal and I. Juhász:* Discrete subspaces of product spaces. General Topology and its Applications (to appear).

[2] *A. Hajnal and I. Juhász:* A consequence of Martin's axiom. Nederl. Akad. Wetensch. Indag. Math. *33* (1971), 457—463.

[3] *Z. Hedrlín:* An application of Ramsey's theorem to the topological products. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. *14* (1966), 25—26.

[4] *E. Hewitt:* A remark on density character. Bull. Amer. Math. Soc. *52* (1946), 641—643.

[5] *I. Juhász:* Cardinal functions in topology. Math. Center Tract No. *34,* Amsterdam, 1971.

[6] *G. Kurepa:* La condition de Souslin et une proprieté characteristique des nombres réels. C. R. Acad. Sci. Paris Sér. A—B *231* (1950), 1113—1114.

[7] *G. Kurepa:* The Cartesian multiplication and the cellularity number. Publ. Inst. Math. (Béograd) *2* (1962), 121—139.

[8] *D. A. Martin and R. M. Solovay:* Internal Cohen extensions. Ann. Math. Logic *2* (1970), 143—178.

[9] *E. S. Pondiczery:* Power problems in abstract spaces. Duke Math. J. *11* (1944), 835—837.