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PROXIMITY APPROACH TO TOPOLOGICAL PROBLEMS

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In this paper we use the techniques of proximity spaces to investigate the properties of semi-metric and developable spaces. By a proximity δ on a nonempty set X, we mean any one of the following: (i) S-proximity, (ii) LO-proximity, (iii) R-proximity, or (iv) EF-proximity (see [6], [7]). Whenever X is a topological space, δ is assumed to be compatible with the topology of X. For a semi-metric space (X, d), we define $\delta(d)$ by: $A\,\delta(d)\,B$ iff $d(A, B) = 0$, where $d(A, B) = \mathrm{Inf}\{d(a, b) : a \in A, b \in B\}$. Set $\mathcal{U}_d = \{U = U^{-1} \subset X \times X : V_{1/n} \subset U$ for some $n \in \mathbf{N}\}$, where $V_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\}$. For a developable space (X, Σ), $\Sigma = \{G_n : n \in \mathbf{N}\}$ where each G_n is an open cover of X and $G_n \supset G_{n+1}$ we set $d(x, y) = \mathrm{Inf}\{1/(n+1) : y \in \mathrm{St}(x, G_n)\}$. The concept of an M-uniformity is defined in [7].

We now present our main results. The following theorem is analogous to the well-known result: A T_1-space has a compatible uniformity with a countable base iff it is metrizable (and consequently has a compatible metric d such that $\delta(d)$ is an EF-proximity).

Theorem 1. *A T_1-space has a compatible* M-*uniformity with a countable base if and only if it has a compatible semi-metric d such that $\delta(d)$ is a* LO-*proximity.*

We give below two new characterizations of developable spaces which can be used to give transparent proofs of some of the results already known in the literature.

Theorem 2. *A T_1-space X is developable if and only if it has a compatible semi-metric d such that $\mathcal{U}_d$ is an* M-*uniformity with a countable open base.*

Theorem 3. (Cf. Cook [5].) *A T_1-space X is developable if and only if it has a compatible upper-semi-continuous semi-metric.* (Cook proved that if the semi-metric is continuous then the space is developable.)

We now present two metrization theorems which are improvements of those of Arkhangel'skii [4] and Nedev [8] respectively.

Theorem 4. *A T_1-space X is metrizable if and only if it has a compatible semi-metric d such that $\delta(d)$ is an* R-*proximity.* (Arkhangel'skii required $\delta(d)$ to be an EF-proximity.)

Theorem 5. *A T_1-space X is metrizable if and only if it has a compatible semi-metric d such that for each closed set $A \subset X$, $d(A, x)$ is a lower-semi-continuous function of x.* (Nedev required $d(A, x)$ to be continuous.)

In the sequel we suppose that δ is a compatible proximity on a T_1-space X and that $f : X \to Y$ is a continuous function onto a T_1-space Y. We generalize the concepts: T_1-map, uniform map, completely uniform map, pseudo-open map etc. so as to include proximity spaces (see [1], [2], [3], [4]). For example f will be called *uniform* iff for each y in Y and each neighbourhood N_y of $y, f^{-1}(y)\ \bar{\delta}\ (X - f^{-1}(N_y))$. When δ is induced by a metric on X, the above definition coincides with the usual one. We also define a nearness relation δ' on Y as follows:

$$E\ \bar{\delta}'\ F \text{ iff } f^{-1}(E)\ \bar{\delta}\ f^{-1}(F)\,.$$

Theorem 6. *δ' is a compatible S-proximity on Y in the following two cases:*

(i) *δ is an S-proximity, f is pseudo-open and uniform;*

(ii) *δ is an R-proximity, f is pseudo-open and compact.*

Theorem 7. *If δ is a LO-proximity on X and f is open uniform, then δ' is a compatible LO-proximity on Y.*

Theorem 8. *Suppose δ is an S-proximity and f is open uniform. Then f is completely uniform if and only if δ' is a compatible R-proximity on Y.*

Theorem 9. *If (X, d) is a semi-metric space and f is open and completely uniform, then Y is metrizable.* (The known result requires d to be a metric [1].)

Theorem 10. *If X is developable and f is open uniform, then Y is developable.*

The detailed paper with proofs will appear in the Pacific Journal of Mathematics.

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