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ON THE IMBEDDING OF EXTREMALLY DISCONNECTED SPACES INTO BICOMPACTA

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1. The formulation of the problem

The class of bicompacta decomposes into two classes $\mathscr{R}_1$ and $\mathscr{R}_2$. The first class includes the bicompacta which contain βN – the Stone-Čech compactification of a countable discrete space – and the second includes the bicompacta which do not contain βN. We note that the class $\mathscr{R}_1$ contains all the infinite quasi-extremal bicompacta (the closure of any open set of the F_σ type is open) and all the dyadic bicompacta of weight $\geqq \mathfrak{c} = \exp \aleph_0$. (A bicompactum X is called dyadic if X is a continuous image of the Cantor space D^τ for some $\tau \geqq \aleph_0$.) The class $\mathscr{R}_2$ contains all the hereditarily normal spaces (and consequently all the linearly ordered bicompacta) and all the sequential bicompacta.

1.1. A necessary and sufficient condition for a bicompactum X to contain βN is that X can be mapped onto a Tyhonov cube $I^{\mathfrak{c}}$ with weight $\mathfrak{c}$.

Note that βN is an extremally disconnected space (the closure of any open set is open). Moreover, βN contains all the extremally disconnected spaces with weight $\leqq \mathfrak{c}$. Naturally there arises a question as to which bicompacta contain extremally disconnected bicompacta having sufficiently large weights?

Let $\tau \geqq \mathfrak{t} \geqq \aleph_0$ and denote $\tau^{1/\mathfrak{t}} = \min \{\mathfrak{m}, \mathfrak{m}^{\mathfrak{t}} \geqq \tau\}$. The cardinal number $\mathfrak{m}$ will be called admissible if $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$.

1.2. Theorem. *If a bicompactum X can be mapped onto a Tyhonov cube I^τ, then X contains all the extremally disconnected spaces with weight $\leqq \tau$. Conversely, if a bicompactum X contains an infinite extremally disconnected bicompactum Y, with weight τ, and Souslin number $\mathfrak{t}$, then X can be mapped onto $I^{\mathfrak{m}}$, $\mathfrak{m} = \tau^{1/\mathfrak{t}} + \mathfrak{k}$ where $\mathfrak{k} = \exp \mathfrak{t}$ if $\mathfrak{t}$ is accessible or else $\mathfrak{k} = \exp \sigma$ for any $\sigma < \mathfrak{t}$, if $\mathfrak{t}$ is a weakly inaccessible cardinal number.*

The last estimate follows from [1].

1.3. Theorem. *Every infinite extremally disconnected bicompactum X with weight τ which satisfies the Souslin condition and for which the cardinal number $\mathfrak{c} + \tau^{1/\aleph_0}$ is $\aleph_0$-admissible, can be mapped onto I^τ.*

1.4. Theorem ([2]). *Any infinite bicompactum X with weight τ and Souslin number* $\mathfrak{t}$, *where* $\tau \geqq \exp\exp\exp \mathfrak{t}$, *contains all extremally disconnected spaces with weight* $\leqq (\exp \tau)^{+}$.

In [2] a special cardinal number invariant called the strength eX of a topological space X is defined.

1.5. Theorem ([2]). *If a bicompactum X can be mapped onto I^{τ}, then $eX \geqq \tau$. Conversely, if $eX > \tau$, then X can be mapped onto $I^{\log(\tau^{+})}$.*

2. The imbedding of extremally disconnected spaces as nowhere dense subsets into dyadic bicompacta and their absolutes

An absolute pX of a topological space X in the Gleason-Ponomarev sense is an irreducible perfect extremally disconnected preimage of X. According to V. I. Ponomarev [3] the class of completely regular spaces decomposes into classes of spaces which are co-absolute to one another. X and Y are called co-absolute if $\mathrm{p}X \underset{\mathrm{top}}{=} \mathrm{p}Y$.

2.1. Theorem. *If a bicompactum X with weight τ and satisfying Souslin's condition can be mapped onto I^{τ}, then there exists a nowhere dense closed $F \subset \mathrm{p}X$ homeomorphic with* pX.

A topological space X is called τ-dispersed if for every closed $F \subset X$ there exists a point $x \in F$ such that $\chi(x, F) < \tau$, where $\chi(x, F)$ denotes the character of the point x in F. If, for example, $\tau = \aleph_0$, then the usual definition of a dispersed space is obtained, i.e. every subset $F \subset X$ is not dense in itself.

2.2. Theorem. *In order that it may be possible to map a dyadic bicompactum X with weight τ onto I^{τ} it is necessary and sufficient that X be not τ-dispersed.*

2.3. Theorem [4]. *The Continuum Hypothesis is equivalent to the following statement: Every non-metrizable dyadic bicompactum contains βN.*

Denote by (γ) the following hypothesis:

$$(\gamma) \quad (\forall\tau)(cf(\tau) = \aleph_0) \,\&\, (\forall\mathfrak{n})(\mathfrak{n} < \tau) \Rightarrow (\exp \mathfrak{n} \leqq \tau)$$

2.4. Theorem. *Let (γ) hold. The absolute of any τ-dispersed dyadic bicompactum X with weight τ can be mapped onto $I^{\exp\tau}$. Moreover, in order that a mapping of* pI^{τ} *onto $I^{\exp\tau}$ may exist it is necessary that $\tau^{\aleph_0} = \exp \tau$ and it is sufficient that $cf(\tau) = \aleph_0$.*

Note that no τ-dispersed dyadic bicompactum with weight τ can be mapped even onto I^{τ}.

2.5. Theorem. *The absolute* pX *of every non-*τ*-dispersed dyadic bicompactum with weight* τ *contains a closed nowhere dense* $F \subset$ pX *homeomorphic with* pX. *An analogous statement is true for the absolutes of* τ*-dispersed dyadic bicompacta provided* (γ) *holds.*

3. Classes of non-homogeneous extremally disconnected bicompacta

A topological space X is called homogeneous if for any two points $x, y \in X$ there exists a homeomorphism $\varphi : X \xrightarrow{\text{onto}} X$ such that $\varphi(x) = y$.

As A. V. Arhangelskii has shown [5] every extremally disconnected bicompactum with weight $\mathfrak{c}$ is non-homogeneous. Z. Frolík [6] proved that if the Continuum Hypothesis is true or if there are cardinal numbers in between $\mathfrak{c}$ and $\exp \mathfrak{c}$ then every infinite extremally disconnected bicompactum is non-homogeneous. The author proved [2] that if cX is weakly accessible and $cX \geqq \log(\pi w X)$, then the extremally disconnected bicompactum X is non-homogeneous. Here we show the non-homogeneity of some new classes of extremally disconnected bicompacta in a number of cases without using any special set theory hypotheses. These results are a consequence of the above theorems and the following Frolík's result [7]: If E is a closed nowhere dense subspace of an extremally disconnected bicompactum X and if E contains X (in particular, if E is homeomorphic to X) then E is non-homogeneous.

3.1. Theorem.[1]) *Any extremally disconnected bicompactum* X *satisfying one of the below conditions is non-homogeneous*:

1) X *satisfies Souslin's condition and the cardinal* $\mathfrak{c} + (wX)^{1/\aleph_0}$ *is* $\aleph_0$*-admissible.*

2) X *is the absolute of the dyadic bicompactum* Y *with weight* τ *where* Y *is not* τ*-dispersed, in particular, if* $cf(wY) \geqq \aleph_1$.

3) X *is the absolute of a* τ*-dispersed bicompactum with weight* τ *provided* (γ) *holds.*

4) X *is the absolute of an ordered bicompactum* Y, wY *being weakly accessible.*

4. The dependence of the power of a bicompactum on its weight

Consider $|X|$ as a function which assigns to each space X of weight $\leqq \tau$ the cardinality of X. If $|X|$ is defined on the set of all infinite metric compacta then it can assume at most two values, viz. $\aleph_0$ or $\mathfrak{c}$. This fact is proved independently of the

[1]) Editor's Note. K. Kunen announced in his preliminary report On the compactification of the integers, Notices Amer. Math. Soc. *17* (1970), p. 299, that the usual orderings on the types of ultrafilters are not linear. Now by another theorem in [6] it follows that no infinite extremally disconnected compact space is homogeneous.

Continuum Hypothesis. If $|X|$ is defined on the set of all bicompacta with weight τ then $\tau \leqq |X| \leqq \exp \tau$. A question naturally arises: Is it again possible to show independently of the Generalized Continuum Hypothesis that the function $|X|$ assumes at most two values? In general, the answer is negative. Namely, in the model M of the set theory ZF in which 1) $\mathfrak{c} = \aleph_{\omega_1}$, 2) $\exp \mathfrak{c} = \aleph_{\omega_2}$, 3) $(\forall \mathfrak{n})(\mathfrak{n} < \mathfrak{c}) \Rightarrow$ $\Rightarrow (\exp \mathfrak{n} < \exp \mathfrak{c})$ there exists a bicompactum with weight $\mathfrak{c}$ for which $\mathfrak{c} < |X| <$ $< \exp \mathfrak{c}$. The existence of such models M is proved by Cohen's method. However, in the case of dyadic bicompacta a positive answer can be given without the GCH or any of its analogues.

4.1. Theorem. *The cardinality of any dyadic bicompactum with weight τ which is not τ-dispersed equals* $\exp \tau$. *The cardinality of any τ-dispersed dyadic bicompactum equals* $\mathfrak{e} = \sum_{k<\omega_0} \exp \mathfrak{n}_k$ *for some countable sequence* $\mathfrak{n}_1 < \mathfrak{n}_2 < \ldots <$ $< \mathfrak{n}_k < \ldots$ *of cardinal numbers, where* $\sum_{k<\omega_0} \mathfrak{n}_k = \tau$.

Note that the cardinal number $\mathfrak{e}$ is independent of the choice of the sequence $\{\mathfrak{n}_k\}$.

4.2. By $\pi w X$ we shall denote the π-weight [3] of a topological space X, i.e. the least cardinal number of a system of open subsets of X cofinal with the system of all open subsets of X ordered according to inclusion.

4.3. Theorem. *The cardinality of every bicompactum X co-absolute with a dyadic bicompactum which is not $(\pi w X)$-dispersed satisfies the inequalities*

$$\exp(\pi w X) \leqq |X| \leqq \exp(w X).$$

and

$$\pi w X \leqq w X \leqq (\pi w X)^{\aleph_0}.$$

If a bicompactum X co-absolute with a dyadic bicompactum is $(\pi w X)$-dispersed then the cardinality of X satisfies the inequality

$$\sum_{k<\omega_0} \exp \mathfrak{n}_k \leqq |X| \leqq \sum_{k<\omega_0} \exp \exp \mathfrak{n}_k$$

for any sequence of cardinal numbers $\mathfrak{n}_1 < \mathfrak{n}_2 < \ldots < \mathfrak{n}_k < \ldots$ such that $\sum_{k<\omega_0} \mathfrak{n}_k =$ $= \pi w X$.

Since any two compactifications of a space X are co-absolute the following Corollary gives a negative answer to A. V. Arhangelskii's question as to whether any countable completely regular space S has a compactification of power $\mathfrak{c}$.

4.4. Corollary. *There exists a countable completely regular space S (for instance a countable dense subset of $I^{\mathfrak{c}}$) such that its every compactification has the power* $\exp \mathfrak{c}$.

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