Harry Poppe

A compactness criterion for Hausdorff admissible (jointly continuous) convergence structures in function spaces

In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 353--357.

Persistent URL: http://dml.cz/dmlcz/700773

A COMPACTNESS CRITERION FOR HAUSDORFF ADMISSIBLE (JOINTLY CONTINUOUS) CONVERGENCE STRUCTURES IN FUNCTION SPACES

H. POPPE

Greifswald

By an L-space $(X, \lim)$ we understand a set X and a mapping lim from the set of all filters of X into the set of all subsets of X which satisfies the following conditions:

(1) For each $x \in X$, $x \in \lim [x]$, where $[x]$ denotes the ultrafilter containing $\{x\}$.

(2) $x \in \lim \psi$ and $\psi \subset \varrho$ implies $x \in \lim \varrho$.

$(X, \lim)$ is called a convergence space and lim a convergence structure for X. $(X, \lim)$ is called an L^*-space, iff lim satisfies:

(3) If ψ is a filter on X and for each ultrafilter $\pi \supset \psi$, $x \in \lim \pi$ holds, then $x \in \lim \psi$.

For $x \in X$, the filter $\mathfrak{U}(x) = \bigcap\{\psi : x \in \lim \psi\}$ is called the neighborhood filter at x. If lim satisfies:

(4) For each $x \in X$, $x \in \lim \mathfrak{U}(x)$,

then $(X, \lim)$ is called a U-space ("Umgebungsraum") or a pretopological space.

$(X, \lim)$ is called a Hausdorff convergence space iff $x \in \lim \psi$ and $y \in \lim \psi$ implies $x = y$, that is for each converging filter ψ, $\lim \psi$ consists of a single element.

In the sequel let X and Y denote L-spaces. By Y^X we understand the set of all functions from X into Y and by $C(X, Y)$ the set of all continuous functions. ω denotes the evaluation map $\omega : Y^X \times X \to Y$, that is, $\omega(f, x) = f(x)$, and a convergence structure lim for Y^X or for $C(X, Y)$ is called *admissible* (*jointly continuous, conjoining*) iff ω: $(Y^X, \lim) \times X \to Y$ is continuous.

A very useful convergence structure for $C(X, Y)$ is that of continuous convergence.

Definition 1. Let $\mathfrak{F}$ be a filter on $C(X, Y)$; $\mathfrak{F}$ is said to *converge continuously to* $f \in C(X, Y)$, $\mathfrak{F} \xrightarrow{c} f$ or $f \in c\text{-}\lim \mathfrak{F}$, iff for each $x \in X$ and each filter ψ, $\psi \to x$ implies $\omega(\mathfrak{F} \times \psi) \to f(x)$, where $\mathfrak{F} \times \psi$ denotes the cartesian product of the filters.

J. L. Kelley and A. P. Morse [1] defined the notion of even continuity, which is a generalization of equicontinuity, for sets of functions from a topological space X into a topological space Y. This notion can be extended to the case of X and Y being only convergence spaces:

Definition 2. Let $H \subset C(X, Y)$, H is called *evenly continuous* iff for each $x \in X$, $y \in Y$, each filter $\mathfrak{F}$ on $C(X, Y)$ such that $H \in \mathfrak{F}$ and each filter ψ on X, $\omega(\mathfrak{F} \times [x]) \to y$ and $\psi \to x$ implies $\omega(\mathfrak{F} \times \psi) \to y$.

Remark. For information about convergence spaces, properties of the convergence structure of continuous convergence and of even continuity see [3], [4], [5] and especially [6].

Proposition 1. *Let $H \subset C(X, Y)$ and $\mathfrak{F}$ be a filter on $C(X, Y)$ such that $H \in \mathfrak{F}$ and $\mathfrak{F}$ converges pointwise to $f \in C(X, Y)$. If H is evenly continuous, then $\mathfrak{F}$ converges to f continuously.*

Proposition 2. *Let* lim *be a convergence structure for $C(X, Y)$. Then* lim *is admissible for $C(X, Y)$ iff* $c\text{-lim} \leqq \text{lim}$, *that is,* $\lim \mathfrak{F} = f$ *implies* $c\text{-lim}\, \mathfrak{F} = f$ *for each filter $\mathfrak{F}$ on $C(X, Y)$.*

Remark. For proofs of Propositions 1 and 2 see [6].

Now we are able to formulate a compactness criterion for a Hausdorff admissible convergence structure lim for $C(X, Y)$. $H \subset C(X, Y)$ is called compact relative to lim iff each ultrafilter on $C(X, Y)$ containing H lim-converges to an element of H.

Theorem 1. *Let X be an L-space and Y a Hausdorff and regular L^*-space and let* lim *be a Hausdorff admissible convergence structure for $C(X, Y)$ (that means, $(C(X, Y), \text{lim})$ is an L-space); let $H \subset C(X, Y)$. The following conditions are necessary and sufficient for the compactness of H relative to* lim:

(a) *H is closed in $C(X, Y)$ relative to* lim.

(b) *$H(x) = \{f(x) : f \in H\}$ is compact for each $x \in X$.*

(c) *H is evenly continuous.*

(d) *If $\mathfrak{F}$ is an ultrafilter on $C(X, Y)$ such that $H \in \mathfrak{F}$ and $\mathfrak{F} \xrightarrow{c} f \in C(X, Y)$, then $f \in \lim \mathfrak{F}$.*

Remark. For the proof of Theorem 1 see [6]. The application of Theorem 1 to particular situations consists in finding conditions which imply condition (d) of Theorem 1. We will illustrate this by two examples.

A) The convergence structure of strictly continuous convergence.

Definition 3. Let $\mathfrak{F}$ be a filter on $C(X, Y)$ (or on Y^X); $\mathfrak{F}$ *converges strictly continuously* to f, $\mathfrak{F} \xrightarrow{str.c} f$ or $f \in str.\ c\text{-lim}\, \mathfrak{F}$, iff for each filter ψ on X the convergence of $f\psi$ to $y \in Y$ implies $\omega(\mathfrak{F} \times \psi) \to y$.

Comparing it with the definition of continuous convergence, we see at once that $c\text{-lim} \leqq str.\ c\text{-lim}$ in $C(X, Y)$ holds, that is, $str.\ c$-lim is admissible for $C(X, Y)$. Moreover, if Y is Hausdorff, $(C(X, Y), str.\ c\text{-lim})$ is Hausdorff, too.

Theorem 2. 1. *Let X be a pretopological space and Y a regular topological space; let $H \subset C(X, Y)$. The following conditions are sufficient for the compactness of H in $C(X, Y)$ relative to str. c-*lim:

(a) *H is closed relative to str. c-*lim.

(b) *$H(x)$ is compact for each $x \in X$.*

(c) *H is evenly continuous.*

(d) *If $\mathfrak{F}$ is an ultrafilter on $C(X, Y)$, $H \in \mathfrak{F}$, π is an ultrafilter on X and $y \in Y$, then $\omega(\mathfrak{F} \times \pi) \to y$ whenever there exists for every neighborhood V of y a set $B_V \in \pi$ with the following property: if $x \in B_V$, then there is a set $H_x \in \mathfrak{F}$, $H_x \subset H$, and a neighborhood U_x of x such that $\omega(H_x \times U_x) \subset V$.*

2. *Let X be a pretopological space and Y a Hausdorff and regular pretopological space. If $H \subset C(X, Y)$ is compact relative to str. c-*lim, *then the conditions* (a), ..., (d) *hold.*

Proof. 1. We show that condition (d) implies the corresponding condition (d) of Theorem 1. Let $\mathfrak{F}$ be an ultrafilter on $C(X, Y)$, containing H and converging continuously to $f \in C(X, Y)$. We must show that $\mathfrak{F}$ converges strictly continuously to f. For this it is sufficient to show that for each ultrafilter π on X, $f\pi \to y$ implies $\omega(\mathfrak{F} \times \pi) \to y$, since Y is an L^*-space. Now let V be an open neighborhood of y; since $f\pi \to y$, there exists $B_V \in \pi$ such that $f(B_V) \subset V$; therefore for $x \in B_V$, V is a neighborhood of $f(x)$; since $f \in c$-lim $\mathfrak{F}$ and $\mathfrak{U}(x) \to x$, we find $A_x \in \mathfrak{F}$ and $U_x \in \mathfrak{U}(x)$ such that $\omega(A_x \times U_x) \subset V$; we have $H_x = A_x \cap H \in \mathfrak{F}$ and $\omega(H_x \times U_x) \subset V$; hence the suppositions of (d) are satisfied and it follows $\omega(\mathfrak{F} \times \pi) \to y$ and hence $f \in$ *str.* c-lim $\mathfrak{F}$. Then the compactness of H relative to *str.* c-lim follows from a c-lim-compactness criterion, which can be found in [3] or [6].

2. If H is compact relative to *str.* c-lim, then, as is easy to see, H is closed relative to *str.* c-lim. Moreover H is compact relative to c-lim, since *str.* c-lim is admissible for $C(X, Y)$. Then conditions (b) and (c) of Theorem 2 follow from the same c-lim-compactness criterion which was mentioned above. We now show that (d) holds. We assume that the suppositions of condition (d) are fulfilled. Let $\mathfrak{F}$ be an ultrafilter on $C(X, Y)$ such that $H \in \mathfrak{F}$, π an ultrafilter on X and $y \in Y$; since H is compact relative to *str.* c-lim, there exists $f \in H$ such that $f \in$ *str.* c-lim $\mathfrak{F}$; we shall show that $f\pi \to y$. Let $V \in \mathfrak{U}(y)$; Y is a regular pretopological space and hence we have $\mathfrak{U}(y) \to y$, which implies $\mathfrak{U}^\lambda(y) \to y$, where the filter $\mathfrak{U}^\lambda(y)$ is generated by $\{U^\lambda : U \in \mathfrak{U}(y)\}$, $U^\lambda = \{y \in Y$: there exists a filter ψ on Y such that $U \in \psi$ and $\psi \to y\}$. We then find $V_1 \in \mathfrak{U}(y)$ such that $V_1^\lambda \subset V$; by the supposition of (d), for V_1 there exists $B_1 \in \pi$ such that for $x \in B_1$ there exist sets $H_x \in \mathfrak{F}$, $H_x \subset H$ and $U_x \in \mathfrak{U}(x)$ such that $\omega(H_x \times U_x) \subset V_1$; we have $f \in c$-lim $\mathfrak{F}$ and therefore $\omega(\mathfrak{F} \times \mathfrak{U}(x)) \to f(x)$, since X is a pretopological space; but $\omega(H_x \times U_x) \subset V_1$ implies $V_1 \in \omega(\mathfrak{F} \times \mathfrak{U}(x))$ and hence $f(x) \in V_1^\lambda \subset V$; thus we have $f(B_1) \subset V$, which implies $f\pi \to y$; since $f \in$ *str.* c-lim $\mathfrak{F}$, it follows that $\omega(\mathfrak{F} \times \pi) \to y$ and hence (d) is shown.

B) A "graph topology" for $C(X, Y)$.

Definition 4. Let X and Y be topological spaces; for $f \in Y^X$ we denote by $\Gamma(f)$ the graph of f, that is, $\Gamma(f) = \{(x, f(x)) : x \in X\} \subset X \times Y$. Let G be an open set in $X \times Y$ and let $(G) = \{f \in C(X, Y) : \Gamma(f) \subset G\}$; then $\{(G) : G \text{ open in } X \times Y\}$ is a basis for a topology for $C(X, Y)$, which we denote by $\tau_{\mathfrak{S}_a}$.

Remark. The topology $\tau_{\mathfrak{S}_a}$ is obtained by the Tychonoff hyperspace topology, restricted to the set of all graphs of the functions from $C(X, Y)$ (see [7]). It was first considered by Naimpally [2]. For a proof of the following proposition see [7].

Proposition 3. *Let X, Y be topological spaces.*

1) *If X is a T_1-space and Y a Hausdorff space, then $(C(X, Y), \tau_{\mathfrak{S}_a})$ is Hausdorff.*

2) *If X is regular, then the $\tau_{\mathfrak{S}_a}$-convergence is finer than the continuous convergence, that is, by Proposition 2 $\tau_{\mathfrak{S}_a}$ is admissible for $C(X, Y)$.*

Theorem 3. *Let X be a regular T_1-space, Y a Hausdorff and regular space; let $H \subset C(X, Y)$.*

The following conditions are necessary and sufficient for the $\tau_{\mathfrak{S}_a}$-compactness of H:

(a) *H is closed in $(C(X, Y), \tau_{\mathfrak{S}_a})$.*

(b) *$H(x)$ is compact for each $x \in X$.*

(c) *H is evenly continuous.*

(d) *Let $\mathfrak{F}$ be an ultrafilter on $C(X, Y)$ such that $H \in \mathfrak{F}$. For each open set $G \subset X \times Y$ such that $pr_X G = X$ there exist systems of open sets in X and in Y, viz. $(U_i)_{i \in I}$, $(V_i)_{i \in I}$, respectively, with the following properties: $(U_i)_{i \in I}$ is a cover of X, $\bigcup_{i \in I}(U_i \times \bar{V}_i) \subset G$, for $i \in I$ there exists $A_i \in \mathfrak{F}$, $A_i \subset H$ such that $\omega(A_i \times U_i) \subset V_i$. Then there exists $B \in \mathfrak{F}$ such that $\Gamma(B) = \{\Gamma(f) : f \in B\} \subset G$.*

Proof. 1. First we show that conditions (a), ..., (d) are sufficient for the $\tau_{\mathfrak{S}_a}$-compactness of H. As in the proof of Theorem 2, for this purpose we only prove that condition (d) implies the corresponding condition (d) of Theorem 1. Let $\mathfrak{F}$ be an ultrafilter on $C(X, Y)$ such that $H \in \mathfrak{F}$ and $f \in c\text{-lim } \mathfrak{F}$; let G be open in $X \times Y$ and $\Gamma(f) \subset G$; for $x \in X$ there exist open sets $\tilde{U}_x$ of X and $\tilde{V}_x$ of Y such that $(x, f(x)) \in \tilde{U}_x \times \tilde{V}_x \subset G$; since Y is regular, for each $x \in X$ there exists an open set V_x such that $f(x) \in V_x \subset \bar{V}_x \subset \tilde{V}_x$; since $f \in c\text{-lim } \mathfrak{F}$, we find an open set U_x and $\tilde{A}_x \in \mathfrak{F}$ such that $x \in U_x \subset \tilde{U}_x$ and $\omega(\tilde{A}_x \times U_x) \subset V_x$. If $A_x = \tilde{A}_x \cap H$, the families $(U_x)_{x \in X}$, $(V_x)_{x \in X}$ fulfill the suppositions of condition (d). Therefore there exists $B \in \mathfrak{F}$ such that $\Gamma(B) = \{\Gamma(f) : f \in B\} \subset G$. But since G is an arbitrary set, this means that $\mathfrak{F} \xrightarrow{\tau_{\mathfrak{S}_a}} f$.

2. We show that the compactness of H relative to $\tau_{\mathfrak{S}_a}$ implies condition (d). Let $\mathfrak{F}$ be an ultrafilter on $C(X, Y)$ such that $H \in \mathfrak{F}$ and let G be an open subset of $X \times Y$ such

that $pr_X G = X$ and G fulfils the suppositions of (d); by the compactness of H there exists $f \in H$ such that $\mathfrak{F} \xrightarrow{\tau_{\mathfrak{S}_a}} f$; we shall prove that $\Gamma(f) \subset G$; then $\mathfrak{F} \xrightarrow{\tau_{\mathfrak{S}_a}} f$ implies the existence of $B \in \mathfrak{F}$ such that $\Gamma(B) \subset G$. For G there exist families of open sets $(U_i)_{i \in I}$ in X and $(V_i)_{i \in I}$ in Y and a family $(A_i)_{i \in I}$ of subsets of H such that $\bigcup_{i \in I} U_i \times \overline{V}_i \subset G$, $\omega(A_i \times U_i) \subset V_i$ for each $i \in I$ and $(U_i)_{i \in I}$ is a covering of X; we assume now that there exists $x_0 \in X$ such that $(x_0, f(x_0)) \notin \bigcup U_i \times \overline{V}_i$; hence we have $x_0 \in U_{i_0}$ and $f(x_0) \notin$ $\notin \overline{V}_{i_0}$ for some index $i_0 \in I$; hence there exists an open set W such that $f(x_0) \in W$ and $W \cap V_{i_0} = \emptyset$; since X is a T_1-space, $G_1 = (X - \{x_0\}) \times Y \cup (U_{i_0} \times W)$ is an open subset of $X \times Y$, containing $\Gamma(f)$, hence there exists $F \in \mathfrak{F}$ such that $\Gamma(F) \subset G_1$; now let g be some element of $F \cap A_{i_0}$; $g \in F$ implies $\Gamma(g) \subset G_1$ and hence $(x_0, g(x_0)) \in$ $\in U_{i_0} \times W$ and therefore we have $g(x_0) \in W$; but on the other hand $g \in A_{i_0}$ and we have $\omega(A_{i_0} \times U_{i_0}) \subset V_{i_0}$ and hence $g(x_0) \in V_{i_0}$, too, which yields a contradiction. Therefore we have $\Gamma(f) \subset \bigcup U_i \times \overline{V}_i \subset G$, as was desired.

References

[1] *J. L. Kelley:* General Topology. Princeton, New Jersey, 1957.

[2] *S. A. Naimpally:* Graph topologies for function spaces. Trans. Amer. Math. Soc. *123* (1966), 267—272.

[3] *H. Poppe:* Stetige Konvergenz und der Satz von Ascoli und Arzelà. Math. Nachr. *30* (1965), 87—122.

[4] *H. Poppe:* Stetige Konvergenz und der Satz von Ascoli und Arzelà II. Monatsb. Deutsch. Akad. Wiss. Berlin *8* (4) (1966), 259—264.

[5] *H. Poppe:* Stetige Konvergenz und der Satz von Ascoli und Arzelà III, IV, V, VI. Proc. Japan Acad. *44* (1968), 234—239, 240—242, 318—321, 322—324.

[6] *H. Poppe:* Compactness in general function spaces. Deutscher Verlag der Wissenschaften zu Berlin (to appear).

[7] *H. Poppe:* Über Graphentopologien für Abbildungsräume I. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. *15* (1967), 71—80.