Franklin D. Tall

Some set-theoretic consistency results in topology

In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 419--425.

Persistent URL: http://dml.cz/dmlcz/700772

SOME SET-THEORETIC CONSISTENCY RESULTS IN TOPOLOGY

F. D. TALL[1])

Toronto

1. Introduction

It has become evident in the past few years that many questions in general topology are independent of the usual set-theoretic axioms (e.g. Zermelo-Fraenkel Set Theory, including the Axiom of Choice). The ramifications of these developments for the study of topology are far from clear. I for one have no intuition as to whether Souslin spaces or separable normal non-metrizable Moore spaces, for example, "really" exist, and therefore cannot say that the undecidability of their existence merely indicates the need for stronger axioms which settle these questions the "right" way. In this note, however, we confine ourselves to mentioning several models of set theory which provide differing answers to various topological questions.

To avoid stating cumbersome relative consistency theorems, we assume the existence of a model of set theory. Also for simplicity, all spaces are assumed to be T_1.

2. Two models of set theory and their topological properties

The two models I know most about exhibit contrasting behavior with respect to problems connected with Souslin's Conjecture (see e.g. [24]) or the Normal Moore Space Conjecture (see e.g. [5]). Let $\mathfrak{A}$ be the result of adjoining to a model of the generalized continuum hypothesis, $\aleph_2$ Cohen subsets of ω_1 (for amplification, see Section 4). Let $\mathfrak{B}$ be any model of *Martin's Axiom* [21] plus $2^{\aleph_0} > \aleph_1$.

In $\mathfrak{A}$, I am kindly informed by T. Jech, Souslin's Conjecture fails, and so there is [20], [19] a space which satisfies the countable chain condition (i.e., every collection of disjoint open sets is countable), although its square does not. Other consequences include the existence of a compact perfectly normal space which is not separable [18], a perfectly normal Lindelöf space with a point-countable base which is not metrizable [4], [22], and a hereditarily separable, normal space which is not Lindelöf [25].

[1]) Some of this material appeared in the author's doctoral dissertation [28], which was supported by NSF grants GP-5913 and GP-8501. The author also acknowledges support from grant A-7354 of the National Research Council of Canada.

In $\mathfrak{B}$, Souslin's Conjecture holds [27], as well as the stronger propositions that the countable chain condition is preserved by arbitrary products (K. Kunen; for a proof see [17, Chapter 5]), and that compact perfectly normal spaces are separable [16]. The status of the other two problems in $\mathfrak{B}$ is open.

There is a separable normal non-metrizable Moore space in $\mathfrak{B}$. (The existence of Example E of [5] follows from a lemma of J. Silver in Section 2.5 of [21] or from Theorem 3.5 of [7] plus Lemmas 8 and 9 of [23].) Consequently [13], [31], there is a metacompact normal Moore space (otherwise known as a normal space with a uniform base [1], or a perfectly normal space with a σ-point-finite base, or a normal space which is the image of a metric space under a continuous open map with compact point inverses; for proofs of these equivalences, see [2], [3], [9], [12], [13]) which is not metrizable. I conjecture there is even a countable chain condition space with these properties in $\mathfrak{B}$. Other topological consequences of Martin's Axiom appear in Chapter 3 of [7], Chapter 5 of [17], [32], and [33].

A separable normal non-metrizable Moore space is, among other things, a normal first countable space containing a closed discrete subspace $Y = \{y_\alpha\}_{\alpha<\omega_1}$ which is not *separated*, i.e., there do not exist mutually disjoint open sets $\{U_\alpha\}_{\alpha<\omega_1}$, $y_\alpha \in U_\alpha$. *In* $\mathfrak{A}$, however, *every closed discrete subspace of cardinality* $\aleph_1$ *in every normal space of character less than* $2^{\aleph_1}$ *is separated.* (The character of a space is the supremum of the local weights – e.g. a first countable space has character $\aleph_0$, which of course is less than $2^{\aleph_1}$.) The proof is sketched in Section 4. Among many consequences (see also [28], [29], [30], [31]) are that countable chain condition normal Moore spaces are metrizable, countable chain condition normal spaces with point-countable bases are Lindelöf, and locally compact, perfectly normal, subparacompact spaces of cardinality $\leqq \aleph_1$ are paracompact. The first, third, and probably the second of these assertions are false in $\mathfrak{B}$.

3. Other models

Many other models have been constructed by set-theorists in the past few years in order to prove various consistency results. To my knowledge, their implications for topology have not been investigated. A fair number of topological problems are of course equivalent to set-theoretic ones, so certain isolated results may be obtained. An example of such equivalence is the following theorem. (The normal case for $\kappa = \aleph_0$, $\lambda = \aleph_1$ is due half to Jones [15], and half to Heath [14].)

Theorem. *Let* $\kappa \leqq \lambda$ *be infinite cardinals. Then* $2^{2^\kappa} \geqq \lambda$ $(2^\kappa \geqq \lambda)$ $(2^\kappa \geqq 2^\lambda)$ *if and only if there is a Hausdorff (resp. regular) (resp. normal) space of density* κ *(the density is the least cardinal of a dense set) containing a closed discrete subspace of cardinality* λ.

Given any "reasonable" (in a well-defined sense) non-decreasing function F mapping the class of cardinal numbers into itself, Easton [10] constructs a model of a set theory in which for all cardinals κ, $2^{\kappa} = F(\kappa)$. Thus *there is a model of set theory, for example, in which no separable normal space contains an uncountable closed discrete subspace, but in which there is a normal space of density* $\aleph_1$ *containing a closed discrete subspace of power* $\aleph_{83}$.

Bukovský [8] considers a model $\mathfrak{D}$ in which $2^{\aleph_0} = 2^{\aleph_1}$ but every uncoutable separable metric space has a subset which is not Borel. In [28] we showed

Theorem. *There is a separable first countable normal space containing an uncountable closed discrete subspace if and only if there is an uncountable separable metric space in which every subset is* F_{σ}.

(Numerous other equivalents may be found in [28] or [29].) Translating, we have that *in* $\mathfrak{D}$, *no first countable separable normal space contains an uncountable closed discrete subspace, but there is a separable normal space containing such a subspace.*

4. Separating closed discrete subspaces in $\mathfrak{A}$

In [28] we proved a general theorem from which it follows that closed discrete subspaces of cardinality $\aleph_1$ in normal spaces of character less than $2^{\aleph_1}$ are separated in $\mathfrak{A}$. This particular case is sufficiently interesting that I think it worthwhile to present here a sketch of the proof which will hopefully be accessible to topologists unacquainted with consistency proofs. We take our basic definitions and theorems concerning forcing from Shoenfield [26].

Definitions. A *notion of forcing* is a partially ordered set C having a largest element. We write $\leqq_C$ for the ordering and 1_C for the largest element, dropping the C when the context is clear. E'ements of C are generally designated by p, q and r. If $p \leqq q$, we say p is an extension of q. A subset D of C is *dense* if every element of C has an extension in D.

Let C be a notion of forcing in a model $\mathfrak{M}$ of set theory. A subset G of C is *C-generic* over $\mathfrak{M}$ (or simply *generic*) if the following conditions hold.

(G 1) $1 \in G$.

(G 2) For all $p \in G$ and $q \geqq p$, $q \in G$.

(G 3) For all $p, q \in G$, p and q have a common extension in G.

(G 4) For all dense sets D in $\mathfrak{M}$, $G \cap D \neq 0$.

For example, let C be the collection of countable partial functions from ω_1 (i.e., functions with domain a countable subset of ω_1) into $\{0, 1\}$ in $\mathfrak{M}$, ordered by $p \leqq q$ if $p \supset q$. Then if G is C-generic over $\mathfrak{M}$, it is easy to verify that

$$G_0 = \{\alpha : p(\alpha) = 0 \text{ for some } p \in G\},$$
$$G_1 = \{\alpha : p(\alpha) = 1 \text{ for some } p \in G\}$$

are disjoint sets whose union is ω_1.

The basic result about generic sets is

Theorem. *Let $\mathfrak{M}$ be a countable model of set theory, and C a notion of forcing in $\mathfrak{M}$. Then there is a set G which is C-generic over $\mathfrak{M}$, and a countable model of set theory $\mathfrak{M}[G]$ which is the smallest model including $\mathfrak{M}$ and containing G.*

Our main result is

Theorem. *Let $\mathfrak{A}_0$ be a model of set theory plus the* GCH *(generalized continuum hypothesis). Let C_{ω_2} be the collection of countable partial functions from $\omega_2 \times \omega_1$ into $\{0, 1\}$ in $\mathfrak{A}_0$, ordered by $p \leqq q$ if $p \supset q$. Let G be C_{ω_2}-generic over $\mathfrak{A}_0$. Then $\mathfrak{A} = \mathfrak{A}_0[G]$ is a model of set theory plus the* GCH *in which every closed discrete subspace of cardinality $\leqq \aleph_1$ in any normal space of character $< 2^{\aleph_1}$ is separated.*

By Gödel [11] and standard arguments, we may assume the existence of a countable model of set theory plus GCH. We also make the usual remark that if desired our theorem can be translated into a relative consistency theorem.

If G is C_{ω_2}-generic over $\mathfrak{A}_0$, $\alpha < \omega_2$, and $G_\alpha = \{\langle \eta, \varepsilon\rangle : \langle \alpha, \eta, \varepsilon\rangle \in \text{some } p \in G\}$, then G_α is a function from ω_1 into $\{0, 1\}$. If $\alpha \neq \beta$, then $G_\alpha \neq G_\beta$. The G_α's – or more precisely the subsets they determine – are known as *Cohen subsets* of ω_1. G can be "recovered" from the G_α's, so we may write $\mathfrak{A}_0[G_\alpha : \alpha < \omega_2] = \mathfrak{A}_0[G]$. We also consider the models $\mathfrak{A}_\beta = \mathfrak{A}_0[G_\alpha : \alpha < \beta]$, $\beta < \omega_2$ (i.e., the smallest model including $\mathfrak{A}_0$ and containing each G_α, $\alpha < \beta$). It can be shown that every member of $\mathfrak{A}_0[G]$ of cardinality $\leqq \aleph_1$ appears in some $\mathfrak{A}_\beta$. It can also be shown that if C_β is the collection of countable partial functions from ω_1 into $\{0, 1\}$ in $\mathfrak{A}_\beta$, then G_β is C_β-generic over $\mathfrak{A}_\beta$.

With these preliminaries, we can state the key lemma and sketch how it is used to get the main theorem.

Lemma. *Let $\mathfrak{M}$ be a model of set theory. Let C be the collection of countable partial functions from ω_1 into $\{0, 1\}$ in $\mathfrak{M}$. Let G be C-generic over $\mathfrak{M}$. In $\mathfrak{M}$, let $\langle X, \mathscr{T}\rangle$ be a topological space, and $Y = \{y_\alpha\}_{\alpha<\omega_1}$ a closed discrete unseparated subspace, such that all its countable subsets are separated. In $\mathfrak{M}[G]$, $\mathscr{T}$ is the basis for a topology $\mathscr{T}(G)$ on X. As noted earlier, G yields disjoint subsets G_0, G_1 of ω_1, and hence disjoint subsets $Y_0 = \{y_\alpha : \alpha \in G_0\}$, $Y_1 = \{y_\alpha : \alpha \in G_1\}$ of Y. Then in $\mathfrak{M}[G]$ there do not exist disjoint open sets about the disjoint closed sets Y_0, Y_1 in the space $\langle X, \mathscr{T}(G)\rangle$.*

Assume Lemma. In $\mathfrak{A}$, let $\langle X, \mathscr{T}\rangle$ be a normal space of character $<2^{\aleph_1}$ and Y a closed discrete subspace of cardinality $\leqq \aleph_1$. We wish to show Y is separated. Since countable closed discrete subspaces of normal – indeed regular – spaces are separated, we may assume Y has cardinality $\aleph_1$. Since for convenience we are assuming GCH in $\mathfrak{A}_0$ and hence in $\mathfrak{A}$, $2^{\aleph_1} = \aleph_2$ in $\mathfrak{A}$. Therefore the character of X is $\leqq \aleph_1$. X and $\mathscr{T}$ may be large, but it is not too difficult to construct another normal space $\langle X', \mathscr{T}'\rangle$ containing Y as a closed discrete subspace with both X' and a basis $\mathscr{B}'$ of cardinality $\aleph_1$, such that Y is separated in $\langle X', \mathscr{T}'\rangle$ if and only if it is separated in $\langle X, \mathscr{T}\rangle$.

There is a $\beta < \omega_2$ such that X', $\mathscr{B}'$, Y are all in $\mathfrak{A}_\beta$. $\mathscr{B}'$ generates a topology $\mathscr{T}_\beta$ on X in $\mathfrak{A}_\beta$. Since $\mathscr{B}'$ is a basis for $\mathscr{T}'$ in $\mathfrak{A}$ and countable subsets of Y are separated there, it follows that countable subsets of Y are separated in $\langle X', \mathscr{T}_\beta\rangle$ in $\mathfrak{A}_\beta$. If Y were unseparated in $\langle X, \mathscr{T}\rangle$ and hence in $\langle X', \mathscr{T}'\rangle$, Y would be unseparated in $\langle X', \mathscr{T}_\beta\rangle$ since $\mathscr{T}'$ and $\mathscr{T}_\beta$ have the same basis. But if Y is unseparated in $\langle X', \mathscr{T}_\beta\rangle$, then by Lemma, in $\mathfrak{A}_\beta[G_\beta] = \mathfrak{A}_{\beta+1}$ there do not exist disjoint open sets in the space $\langle X', \mathscr{T}_\beta(G_\beta)\rangle$ about $Y_{\beta 0}$, $Y_{\beta 1}$.

To conclude that $\langle X', \mathscr{T}'\rangle$ is not normal, a contradiction, it only remains to verify that the adjunction of $\{G_\alpha\}_{\beta<\alpha<\omega_2}$ does not undo the destruction of normality. The proof, due to K. Kunen, unfortunately must be omitted since it would take too long to provide the necessary background in forcing. The proof of the key lemma cannot be given here for the same reason, but what we can demonstrate – modulo some details – is the weaker fact that in $\mathfrak{M}[G]$ there do not exist disjoint members of $\mathscr{T}$ about Y_0, Y_1.

Let $\mathfrak{M}$, C, G, X, $\mathscr{T}$, Y be as in the hypothesis of Lemma. Let $U_0 \supset Y_0$, $U_1 \supset Y_1$, U_0, $U_1 \in \mathscr{T}$. Then for each $y_\alpha \in Y$, there is a $V_\alpha \in \mathscr{T}$ containing y_α and included in U_0 or U_1, according to whether $y_\alpha \in Y_0$ or Y_1.

For $p \in C$, let $p_0 = \{\alpha : p(\alpha) = 0\}$, $p_1 = \{\alpha : p(\alpha) = 1\}$. Claim

$$D = \{p : \bigcup\{V_\alpha : \alpha \in p_0\} \cap \bigcup\{V_\alpha : \alpha \in p_1\} \neq 0\}$$

is dense. Once this is proved, we are done. D is a dense subset of C in $\mathfrak{M}$ so there is a $p \in G \cap D$. By definition $G_0 \supset p_0$, $G_1 \supset p_1$, so

$$\bigcup\{V_\alpha : \alpha \in G_0\} \cap \bigcup\{V_\alpha : \alpha \in G_1\} \neq 0\,,$$

and hence $U_0 \cap U_1 \neq 0$.

To see that D is dense, let p be an arbitrary member of C. Domain p is countable, so by hypothesis there exist open mutually disjoint W_α, $\alpha \in$ domain p, $y_\alpha \in W_\alpha \subset V_\alpha$. Also by hypothesis the collection

$$\{W_\alpha\}_{\alpha\in \text{domain } p} \cup \{V_\alpha\}_{\alpha\notin \text{domain } p}$$

does not separate Y, so there is an $\alpha_1 \notin$ domain p, and an α_2, such that $V_{\alpha_1} \cap V_{\alpha_2} \neq 0$.

If $\alpha_2 \notin$ domain p, define $q \in C$ by

$$q = p \cup \{\langle \alpha_1, 1 \rangle\} \cup \{\langle \alpha_2, 0 \rangle\} .$$

If $\alpha_2 \in$ domain p, say $p(\alpha_2) = \varepsilon$, define

$$q = p \cup \{\langle \alpha_1, 1 - \varepsilon \rangle\} .$$

In either case $q \leqq p$ and $q \in D$.

A clever forcing argument due to J. Silver plus a fact about locally countable covers, reduce Lemma to the case we have just considered. All details can be found in [28], as can remarks about extending Theorem.

The character restriction entered into the proof in a natural fashion. It is therefore not surprising that it is best possible. Example G of [5] is a normal space of character $2^{2^{\aleph_0}}$ containing a closed discrete unseparated subspace of cardinality $\aleph_1$.

References

[1] *P. S. Alexandrov:* On the metrisation of topological spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. *8* (1960), 135—140.

[2] *A. Arhangel'skiĭ:* On mappings of metric spaces. Soviet Math. Dokl. *3* (1962), 953—956.

[3] *C. E. Aull:* Topological spaces with a σ-point-finite base. Proc. Amer. Math. Soc. *29* (1971), 411—416.

[4] *H. R. Bennett:* Quasi-developable spaces. Topology Conference, Arizona State University, 1967, Tempe, Arizona, 1968, 314—317.

[5] *R. H. Bing:* Metrization of topological spaces. Canad. J. Math. *3* (1951), 175—186.

[6] *R. H. Bing:* Challenging conjectures. Amer. Math. Monthly *74* (1967), 56—64.

[7] *D. D. Booth:* Countably indexed ultrafilters. Thesis, University of Wisconsin, Madison, 1969.

[8] *L. Bukovský:* Borel subsets of metric separable spaces. General Topology and its Relations to Modern Analysis and Algebra, II (Proc. Second Prague Topological Sympos., 1966). Academia, Prague, and Academic Press, New York, 1967, 83—86.

[9] *M. M. Čoban:* On σ-paracompact spaces. Vestnik Moskov. Univ. Ser. I Mat. Meh. *24* (1) (1969), 20—27.

[10] *W. B. Easton:* Powers of regular cardinals. Ann. Math. Logic *1* (1970), 139—178.

[11] *K. Gödel:* Consistency-proof for the generalized continuum hypothesis. Proc. Nat. Acad. Sci. U.S.A. *25* (1939), 220—224.

[12] *S. Hanai:* On open mappings, II. Proc. Japan Acad. *37* (1961), 233—238.

[13] *R. W. Heath:* Screenability, pointwise paracompactness and metrization of Moore spaces. Canad. J. Math. *16* (1964), 763—770.

[14] *R. W. Heath:* Separability and $\aleph_1$-compactness. Colloq. Math. *12* (1964), 11—14.

[15] *F. B. Jones:* Concerning normal and completely normal spaces. Bull. Amer. Math. Soc. *43* (1937), 671—677.

[16] *I. Juhász:* Martin's Axiom solves Ponomarev's problem. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. *18* (1970), 71—74.

[17] *I. Juhász:* Cardinal functions in topology. Mathematical Center Tract *34*, Mathematisch Centrum, Amsterdam, 1971.

[18] *D. Kurepa:* Le problème de Suslin et les éspaces abstraits. C. R. Acad. Sci. Paris Sér. A—B *204* (1937), 325—327.

[19] *D. Kurepa:* Sur une propriété caractéristique du continu linéaire et le probléme de Souslin. Publ. Inst. Math. (Beograd) *4* (1952), 97—108.

[20] *G. Kurepa:* La condition de Souslin et une propriété caractéristique des nombers réels. C. R. Acad. Sci. Paris Sér. A—B *231* (1950), 1113—1114.

[21] *D. Martin and R. M. Solovay:* Internal Cohen extensions. Ann. Math. Logic *2* (1970), 143—178.

[22] *V. I. Ponomarev:* Metrizability of a finally compact p-space with a point countable base. Soviet Math. Dokl. *8* (1967), 765—768.

[23] *F. Rothberger:* On some problems of Hausdorff and of Sierpiński. Fund. Math. *35* (1948), 29—46.

[24] *M. E. Rudin:* Souslin's conjecture. Amer. Math. Monthly *76* (1969), 1113—1119.

[25] *M. E. Rudin:* A normal hereditarily separable non-Lindelöf space. Illinois J. Math. (to appear).

[26] *J. R. Shoenfield:* Unramified forcing. Proc. Sympos. Pure Math. 13, part I, American Mathematical Society, Providence, Rhode Island, 1971, 357—382.

[27] *R. M. Solovay and S. Tennenbaum:* Iterated Cohen extensions and Souslin's problem. Ann. of Math. *94* (1971), 201—245.

[28] *F. D. Tall:* Set-theoretic consistency results and topological theorems concerning the normal Moore space conjecture and related problems. Thesis, University of Wisconsin, Madison, 1969.

[29] *F. D. Tall:* New results on the normal Moore space problem. Proceedings of the Washington State University Conference on General Topology, March 1970, Pullman, Washington, 1970, 120—125.

[30] *F. D. Tall:* A set-theoretic proposition implying the metrizability of normal Moore spaces. Proc. Amer. Math. Soc. *33* (1972), 195—198.

[31] *F. D. Tall:* On the existence of normal metacompact Moore spaces (normal Hausdorff spaces with uniform bases) which are not metrizable. Canad. J. Math. (to appear).

[32] *F. D. Tall:* Souslin's conjecture revisited. Proc. Bolyai János Math. Soc. Colloquium on Topology, Keszthely, Hungary, June 19—23, 1972 (to appear).

[33] *F. D. Tall:* The countable chain condition vs. separability — applications of Martin's Axiom.