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In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 93--95.

Persistent URL: http://dml.cz/dmlcz/700771

H-CLOSED EXTENSIONS OF TOPOLOGICAL SPACES

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P. S. Alexandroff and P. S. Urysohn [1] introduced the following

Definition 1. A Hausdorff (or T_2) space E is *H-closed* if it is closed in every T_2-space E' in which it is contained.

The same authors gave the following characterization of H-closed spaces:

Theorem 1. (Alexandroff-Urysohn) *A T_2-space E is H-closed if and only if from every open covering $E = \bigcup_{i\in I} G_i$ a finite system can be selected such that $E = \bigcup_{j=1}^{n} \bar{G}_{i_j}$.*

The latter condition may be formulated for any topological space, whether T_2 or not, and is fulfilled in particular for every compact space. Therefore, let us introduce

Definition 2. A topological space E is *almost compact* if in each open covering $E = \bigcup_{i\in I} G_i$ there is a finite subsystem $G_{i_1}, \ldots, G_{i_n}$ such that $E = \bigcup_{j=1}^{n} \bar{G}_{i_j}$.

The almost compact spaces have many properties analogous to those of compact spaces. We mention only the following ones:

A topological space E is almost compact if and only if

1) every open filter has a cluster point, or

2) every maximal open filter is convergent (a filter is called open if it has an open base).

Every almost compact regular space is compact.

We see from Theorem 1 that almost compact spaces are generalizations of H-closed T_2-spaces.

However, it is interesting to formulate a direct generalization of the definition of H-closed T_2-spaces, equivalent to the condition of almost compactness. This may be done by means of

Definition 3. ([2]) Let E' be a topological space and $E \subset E'$ a subspace of E'. The space E' is said to be T_2 *with respect to* E if arbitrary two points $x \in E' - E$, $x \neq y \in E'$ have disjoint neighbourhoods.

We can now formulate the definition of H-closedness for a topological space, whether T_2 or not:

Definition 4. A topological space E is *H-closed* if it is closed in every space $E' \supset E$, E' being T_2 with respect to E.

It is easy to see that, if E is T_2, Definition 4 and Definition 1 are equivalent. Moreover, Theorem 1 can be generalized as follows:

Theorem 2. *A topological space E is almost compact if and only if it is H-closed.*

Concerning the extensions of a topological space E, Alexandroff and Urysohn asked whether every T_2-space E has an extension E' which is T_2 and H-closed. M. H. Stone [3] gave a positive answer to this question. Since then, a number of authors: A. D. Alexandroff [4], S. Fomin [5], N. Shanin [6], M. Katětov [7], [8], J. Flachsmeyer [9] etc. have investigated H-closed T_2-extensions of T_2-spaces.

A direct generalization of the problem of H-closed T_2-extensions of T_2-spaces would be the question whether an arbitrary topological space has H-closed extensions. However, the question is obvious in this form, because each topological space possesses e.g. compact extensions. In order to formulate an adequate generalization of the problem of H-closed T_2-extensions of T_2-spaces, we need the following

Definition 5. E' is an *ordinary extension* of the topological space E if it is T_2 with respect to E.

Now we look for *ordinary H-closed extensions* of a topological space E.

If E itself is T_2, an ordinary H-closed extension is the same as an H-closed T_2-extension. It turns out that the theory of ordinary H-closed extensions is very similar to that of H-closed T_2-extensions. E.g., the construction of Flachsmeyer [9] may be transferred with slight modifications to general topological spaces and permits to construct a number of ordinary extensions. For this purpose, let $\mathfrak{P}$ be a base in E such that $\mathfrak{P}$ is a lattice and $P \in \mathfrak{P}$ implies $E - \bar{P} \in \mathfrak{P}$. A filter in E is said to be a $\mathfrak{P}$-filter if it has a base composed of sets belonging to $\mathfrak{P}$. Let us take a set $E' \supset E$ such that there exists a one-to-one map $\mathfrak{S}$ from $E' - E$ onto the set of all non-convergent maximal $\mathfrak{P}$-filters. Further, for $x \in E$ let us denote by $\mathfrak{S}(x)$ the neighbourhood filter of x in E. Now, there exist topologies on E' such that the trace in E of the neighbourhood filter of $x \in E'$ coincides with $\mathfrak{S}(x)$. Among these topologies, there is the coarsest one denoted by $\sigma(\mathfrak{P})$ and the finest one denoted by $\tau(\mathfrak{P})$. The set E', equipped with either $\sigma(\mathfrak{P})$ or $\tau(\mathfrak{P})$ or an arbitrary topology between $\sigma(\mathfrak{P})$ and $\tau(\mathfrak{P})$ is an ordinary H-closed extension of E.

The above construction is far from yielding all possible ordinary H-closed extensions. However, it furnishes a lot of important ordinary H-closed extensions. E.g. E', equipped with $\sigma(\mathfrak{P})$, is an ordinary H-closed extension having a base such that the boundary of its elements is contained in E, and conversely, each extension of this kind is obtained by this construction. In particular, if $\mathfrak{P} = \mathfrak{G}$ (the system of all open sets of E), then E' equipped with $\sigma(\mathfrak{G})$ is the ***Fomin extension*** of E. It is characterized by the properties of being an ordinary H-closed strict and hypercombinatorial extension; by a strict extension of E, we understand an extension E' such that the closures of subsets of E constitute a base for the closed sets in E', and E' is a hypercombinatorial extension if $\bar{A} \cap \bar{B} = A \cap B$ whenever A and B are closed and $A \cap B$ is nowhere dense in E.

Another important particular case is E' equipped with $\tau(\mathfrak{G})$, called the *Katětov extension* of E. It is characterized by being an ordinary H-closed, hypercombinatorial extension such that $E' - E$ is a discrete closed subset of E'.

It can be shown that the Katětov extension E' is the *finest* ordinary H-closed extension of the given space E in the sense that an arbitrary ordinary H-closed extension of E is a continuous image of E' under a map coinciding on E with the identity.

With the help of Flachsmeyer's method we can examine other types of H-closed extensions too. E.g., if E is *semi-regular* (it possesses a base, composed of interiors of closed sets), then it has a semi-regular ordinary H-closed extension, namely E' equipped with $\sigma(\mathfrak{P})$ where $\mathfrak{P}$ is the system of the interiors of all closed subsets of E.

Finally, let us mention an open question:

Which spaces E have an ordinary compactification E'?

If E is a T_2-space, then it has to be completely regular (a Tychonoff space).

If E is not T_2, a necessary condition is that in E the closure of a compact set has to be compact. A sufficient condition is that E is compact, or that each point of E has a compact closed neighbourhood. However, I do not know a necessary and sufficient condition.

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