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ON A CLOSED RANGE THEOREM FOR NONLINEAR OPERATORS

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Let X and Y be Banach spaces, T a bounded linear operator from X to Y and T^* its conjugate from Y^* to X^*. It can be shown that the range of T is closed if and only if it is the set of all y in Y for which $\langle y, y^* \rangle = 0$ for y^* in ker T^*. The operator T is called normally solvable if, for y in Y, the equation $Tx = y$ has a solution if and only if $y \in (\ker T^*)^{\perp}$. Then the closed range theorem is equivalent to the statement that the operator T is normally solvable if and only if $T(X)$ is closed in Y.

When T is nonlinear and Fréchet differentiable it is possible to obtain closed range theorems by defining normal solvability of T for suitably restricted X and Y. In [3] S. I. Pohožaev considers a uniformly convex Y and defines T to be normally solvable when

(i) for any y in Y, there is a sequence $\{y_n\}$ such that $y_n \to y$ and for every y_n there exists $x_n \in X$ minimizing the functional $\|Tx - y_n\|$, and

(ii) for any such sequence $\{y_n\}$ if $T(x_n) - y_n \in [\ker T'(x_n)^*]^{\perp}$ then $y \in T(X)$.

His result can be stated in the following form: Let X be a Banach space and Y a Banach space which admits nearest points, i.e., for each closed set M in Y, the set of all x in Y, for which there is a y in M with $\|x - y\| = d(x, M)$, is dense in Y. Let T be a possibly nonlinear Fréchet differentiable operator from X to Y. The operator T is normally solvable if and only if the range $T(X)$ is closed in Y.

D. E. Wulbert [4] has shown that, besides uniformly convex Banach spaces, the following two classes of Banach spaces admit nearest points: (a) $2R$ Banach spaces or 2-fully convex Banach spaces of Ky Fan and I. Glicksberg [see 2, p. 113]. X is defined to be such a space when if $\{x_n\}$ is a sequence in X such that $\|x_n\| = 1$ for every n, and $\|x_m + x_n\| \to 2$ as $m, n \to \infty$, then $\{x_n\}$ is a Cauchy sequence. (b) Uniformly smooth [see 2, p. 113] Banach spaces satisfying the property that if a sequence $\{x_n\}$ converges weakly to x and if $\|x_n\| \to \|x\|$, then x_n converges strongly to x. Since there exist $2R$ Banach spaces which are not isomorphic to a uniformly convex space we have a positive answer to the question raised by S. I. Pohožaev in [3]. It should be pointed out that F. E. Browder [1] has formulated the underlying theory in a very elegant setting by considerably sharpening and generalizing the result to X a locally convex space and Y any Banach space.

References

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