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THREE CLASSES OF UNIFORM SPACES

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1. Introduction

The classes consist of separable uniform spaces which are, respectively, subfine, $\mathcal{M}$-fine, and measurable (defined below). They have enough in common, and there are enough interesting relationships between them, to warrant discussing them together.

The results on subfine spaces and duality are due largely to Isbell and collaborators, with embellishments in [4d]. $\mathcal{M}$-fine and measurable spaces are treated in [4b, c]. (For other references, see these three papers.) There is some overlap with recent results of H. Gordon. We shall mention several unpublished results of M. D. Rice [7]. I cannot over-emphasize the dependence of this paper on Isbell's work, particularly [5a] and the paper [3] with Ginsberg.

Definitions. A fine uniform space αX is a uniformizable topological space X with its finest compatible uniformity α. If μX is a uniform space, $T\mu X$ is the associated topological space. μX is subfine if it is a subspace of a fine space: $\mu X = \alpha Y|X$. Let $\mathcal{M}$ be the class of metric spaces. μX is $\mathcal{M}$-fine if uniform continuity of $\mu X \xrightarrow{f}$ $\xrightarrow{f} \varrho M \in \mathcal{M}$ implies that of $\mu X \xrightarrow{f} \alpha T\varrho M$. μX is measurable if there is a σ-field $\mathcal{F} \subset 2^X$ such that the countable $\mathcal{F}$-covers (covers of X with members in $\mathcal{F}$) are a basis for μ; these are so-named because, if μX and νY are two, associated with σ-fields $\mathcal{F}$ and $\mathcal{G}$ then $\mu X \xrightarrow{f} \nu Y$ is uniformly continuous iff $f^{-}(G) \in \mathcal{F}$ for each $G \in \mathcal{G}$, i.e., f is "measurable". μX is called separable if μ has a basis of countable covers; equivalently, μ is weak generated by uniformly continuous maps to separable metric spaces.

2. Coreflective subcategories

Subcategory $\mathcal{D}$ of $\mathcal{A}$ is coreflective in $\mathcal{A}$ if to each object $A \in \mathcal{A}$ is associated $dA \in \mathcal{D}$ and map $dA \xrightarrow{i} A$ with all maps $\mathcal{D} \ni D \xrightarrow{f} A$ factoring uniquely, as $f = i \circ g$. Then d is functorial, called the coreflector. Examples: fine spaces in uniform spaces, with coreflector αT; "Shirota spaces" (i.e., separable reflections $e\alpha X$ of fine spaces) in separable spaces, with coreflector $e\alpha T$. Propositions (Kennison [6], Frolík [2]): if $\mathcal{A}$ is the category of (separable) uniform spaces, $\mathcal{D}$ is coreflective iff $\mathcal{D}$ is closed under quotients and sums $\sum$ $(e\sum)$; and, coreflection maps i are one-to-one, and $d\mu X$ can be interpreted as the coarsest uniformity on X which is in $\mathscr{D}$ and finer than μ.

Let $\mathscr{U}$ be a class of uniform spaces, and o a "pre-coreflector", of $\mathscr{U}$ into uniform spaces, with map $oU \xrightarrow{1} U$. μX is called a $\mathscr{U}$-o space if maps $\mu X \xrightarrow{f} U \in \mathscr{U}$ factor uniquely, as $f = 1 \circ g$; that is, if uniform continuity of $\mu X \xrightarrow{f} U$ implies that of $\mu X \xrightarrow{f} oU$. If $o = \alpha T$, we say "$\mathscr{U}$-fine".

We shall consider $\mathscr{U}$-fine spaces, where $\mathscr{U}$ is $\mathscr{M}$, $\gamma\mathscr{M}$ = complete metric spaces, $\mathscr{I}$ = injective uniform spaces, $\{\varrho\mathscr{R}^{\aleph_0}\}$ where $\varrho\mathscr{R}^{\aleph_0}$ is the product of $\aleph_0$ real lines $\varrho\mathscr{R}$; and $\mathscr{U}$-b spaces, where $\mathscr{U}$ is either sep $\mathscr{M}$ or $\{\varrho\mathscr{R}\}$, and for $\varrho M \in$ sep $\mathscr{M}$, $b\varrho M$ is the measurable space associated with the Baire sets of M — note that $b\varrho \supset \alpha T\varrho$.

2.1. Theorem. *In general, $\mathscr{U}$-o spaces form a coreflective subcategory of uniform spaces, with coreflector extending o. The separable $\mathscr{U}$-o spaces form a coreflective subcategory of separable spaces (with coreflector extending $o/$sep $\mathscr{U}$) iff the coreflector preserves separability (equivalently, e preserves the $\mathscr{U}$-o property); this applies to the examples above.*

2.2. Theorem. *Subfine = $\mathscr{I}$-fine* (Isbell, Rice); *for separable spaces, subfine = $\gamma\mathscr{M}$-fine* (Isbell, Ginsburg, Corson); *separable subfine = $\{\varrho\mathscr{R}^{\aleph_0}\}$-fine and $\varrho\mathscr{R}^{\aleph_0}$-weak. Hence separable $\mathscr{M}$-fine $\Rightarrow$ subfine.*

For separable spaces, measurable = sep $\mathscr{M}$-b = $\{\varrho\mathscr{R}\}$-b. Hence, measurable $\Rightarrow$ $\Rightarrow$ $\mathscr{M}$-fine.

A non-separable $\mathscr{M}$-fine, or $\gamma\mathscr{M}$-fine, space need not be subfine. Concerning measurability in the non-separable case, the proper *definition* has not been found; this was noted also by Frolík in his lecture.

2.3. Theorem. *If all spaces in $o\mathscr{U}$ are complete, then the completion of a $\mathscr{U}$-o space is $\mathscr{U}$-o.*

We use 2.2 to apply this to: subfine, because injective spaces are complete; $\mathscr{M}$-fine because metric spaces are topologically complete; measurable because $b\varrho\mathscr{R}$ is complete (Marczewski-Sikorski). Rice has shown that if coreflective $\mathscr{D}$ has topology-preserving coreflector, then $D \in \mathscr{D}$ implies the completion $\gamma D \in \mathscr{D}$; an example shows the restriction is needed. This applies to subfine, $\mathscr{M}$-fine, and fine (which 2.3 does not), but not to measurable.

3. The functors

By § 2, subfine and $\mathscr{M}$-fine are coreflective in all uniform spaces, and measurable spaces in separable spaces; there are coreflectors l, m, b. Isbell describes l by embedding $\mu X \to I \in \mathscr{I}$ (injective), and taking the subspace $\alpha TI/X$. This section concerns concrete descriptions of b, and m on separable spaces. (More generally, Rice has described a $\mathscr{U}$-o coreflector by a transfinite process similar to that used in [3] for the "locally-fine" coreflector — which agrees with l in separable spaces.) In what follows, we shall occasionally refer to a general coreflective subcategory $\mathscr{D}$, with coreflector d.

Each space is supposed separable. coz $C(\mu X)$ is the set of all cozero sets of real-valued uniformly continuous functions; $\mathscr{Z}(C(\mu X))$ consists of the complements. If $\mathscr{A} \subset 2^X$, $\sigma(\mathscr{A})$ is the generated σ-field.

3.1. Proposition. *μX is ($\mathscr{M}$-fine; measurable) iff each countable* (coz $C(\mu X)$; $\sigma(\text{coz } C(\mu X))$)*-cover is in μ. Hence, for general μX: ($m\mu X$; $b\mu X$) has basis of all countable* (coz $C(\mu X)$; $\sigma(\text{coz } C(\mu X))$)*-covers.*

Real-valued functions. How are $C(\mu X)$ and $C(d\mu X)$ related? We describe this for m and b, using 3.1; with no analogue for l, we have no description of $C(l\mu X)$.

3.2. Theorem. (a) *$f \in C(m\mu X)$ iff $f^{-1}(G) \in \text{coz } C(\mu X)$, for open $G \subset \mathscr{R}$.*

(b) *$C(m\mu X)$ is the uniform closure of $\{f/g : f, g \in C^*(\mu X), g \text{ never } 0\}$.*

(c) *If $f \in C(m\mu X)$ and $f \geqq 0$, then there is a sequence (f_n) from $C(\mu X)$ with $f_n \uparrow f$ pointwise.*

3.3. Theorem. (a) *$f \in C(b\mu X)$ iff $f^{-1}(G) \in \sigma(\text{coz } C(\mu X))$, for open $G \subset \mathscr{R}$.*

(b) *$C^*(b\mu X)$ is the uniform closure of the $\sigma(\text{coz } C(\mu X))$ – simple functions.*

(c) *$C(b\mu X)$ is the least class containing $C(\mu X)$ and closed under pointwise convergence of sequences.*

The approximation theorems 3.2 (b) and (c) apply to all continuous functions when $m\mu X = \alpha T\mu X$. This occurs if $T\mu X$ is Lindelöf (below).

3.3 can be proved from results of Mauldin. 3.2 (c) shows that $C(m\mu X)$ is a subset of $\mathscr{B}_1$, the first Baire class of $C(\mu X)$. Maulding shows that $f \in \mathscr{B}_1$, iff $f^{-1}(G) \in \in \mathscr{Z}(C(\mu X))_\sigma$, for open $G \subset \mathscr{R}$; compare 3.2 (a).

Topology. It is easily seen that l, m, preserve topology. b does not: $Tb\mu X$ carries the coarsest P-space topology (G_δ's are open) finer than $T\mu$, because $\sigma(\text{coz } C(\mu X))$ is an open basis. $\mathscr{Z}(C(\mu X))$ is another, and $\mathscr{B}_1$ generates $Tb\mu$.

When does uniformizable X have unique uniformity in $\mathscr{D}$? Equivalently if fine spaces $\in \mathscr{D}$, when is $d\mu X = \alpha T\mu X$? Since precompact spaces are subfine, X has unique subfine uniformity iff X has unique compactification (or uniformity) (Doss-Hewitt): X is "almost compact" = of each pair of disjoint zero sets, one is compact. For m: X has unique $\mathscr{M}$-fine uniformity iff X is Lindelöf or almost compact (essentially, Henrikson-Johnson and Hager-Johnson). For b: X has unique measurable uniformity iff X is "almost Lindelöf" and P (essentially Frolík).

Subspaces. When is $d(\mu Y/X) = d\mu Y/X$? Rice has some results for $\mathscr{D} = \mathscr{U}\text{-}o$, but any general answer is far from clear. For l, the equation holds, as Isbell shows from his contruction of l [5b]. Likewise for b; this is the equality $\sigma(\text{coz } C(\mu Y/X)) = = \sigma(\text{coz } C(\mu Y) \cap X)$, and uses the Katětov extension theorem. For m, we have the following rather complicated result, with interesting corollaries.

3.4. Theorem. *$m(\mu Y/X) = m\mu Y/X$ iff $Z \in \mathscr{Z}(C(\mu Y))$ and $Z \cap X = \emptyset$ imply a $Z_1 \in \mathscr{Z}(C(\mu Y))$ with $Z_1 \supset X$ and $Z_1 \cap Z = \emptyset$.*

Taking μY a Shirota space $e\alpha Y$, 3.4 becomes: $e\alpha Y/X$ is $\mathcal{M}$-fine iff X is completely separated from every disjoint zero set. Now, it is easy to see that $m(e\alpha Y/X) = e\alpha X$ iff $\mathscr{Z}(C(X)) = \mathscr{Z}(C(Y)) \cap X$, i.e., X is "Z-embedded". Then, we get the equivalence of (a) $e\alpha Y/X = e\alpha X$; (b) X is Z-embedded and completely separated from every disjoint zero set; (c) X is C-embedded, i.e., $C(X) = C(Y)/X$. (a) $\Leftrightarrow$ (c) is due to Shapiro-Gantner; (b) $\Leftrightarrow$ (c) improves and clarifies a result of Gillman-Jerison.

In case X is dense in Y, the condition in 3.4 becomes: each G_δ in Y meets X, or, X is "G_δ-dense".

3.5. Corollary. *Let μX be separable subfine, $= e\alpha Y/X$. Then, μX is $\mathcal{M}$-fine iff X is G_δ-dense in $\overline{X}^Y$. Hence, if μX is precompact, μX is $\mathcal{M}$-fine iff X is G_δ-dense in the compactification (completion).*

3.6. Corollary (of 3.4). *μX is measurable iff μX is separable and hereditarily $\mathcal{M}$-fine.*

Completion. When is $d\gamma = \gamma d$? (γ is the completion functor.) Isbell observes that this holds if d preserves topology and subspaces, and that this applies to l [5b]. Neither m nor b preserves both.

3.7. Lemma. *The points of $\gamma m\mu X$ "are" the $\mathscr{Z}(C(\mu X))$-ultrafilters with cip; these are in one-to-one correspondence with the $\sigma(\mathrm{coz}\ C(\mu X))$-ultrafilters with cip. Hence, $\gamma m\mu X$ and $\gamma b\mu X$ live on the same set; and $b\mu X$ is complete iff $m\mu X$ is (which occurs if μX is).*

3.8. Lemma. *$\gamma b\mu X$ is a subspace of $b\gamma\mu X$, living on the closure of X in $b\gamma\mu X$ (in the $b\gamma\mu$-topology); this is the G_δ-closure of X in $\gamma\mu X$.*

3.9. Theorem. *$m\gamma\mu X = \gamma m\mu X$, and $b\gamma\mu X = \gamma b\mu X$, iff X is G_δ-dense in $\gamma\mu X$.*

This also uses 3.4 and 2.3. 3.8 uses the results for b on topology and subspaces. The first part of 3.7 is rather standard, generalizing Hewitt and Shirota on υX vs. $e\alpha X$. The second part of 3.7 follows from results of Hays and Frolík generalizing Hewitt and Marczewski-Sikorski on realcompactness vs. completeness of $be\alpha X$.

4. Functions algebras and duality

Consider a family A: $X \to R$ of real-valued functions on X which separates the points of X, is an algebra and lattice in the pointwise operations, is closed under uniform convergence, and contains the constant 1. A is: closed under countable composition (cc) if $f_1, f_2, \ldots \in A$ and $f \in C(R^{\aleph_0})$ imply that $f \circ (f_i) \in A$; closed under inversion (ci) if $f \in A$ and f never 0 imply that $1/f \in A$; regular if $f \in A$ implies $g \in A$ with $f^2 g = f$. Then, regular $\Rightarrow$ ci $\Rightarrow$ cc; the latter is not obvious. Let $\mathscr{A}_i(X)$, $i = 1, 2, 3$, be the class of A: $X \to R$ which are, respectively, cc, ci, and regular. Let $\mathscr{S}_i$, $i = 1, 2, 3$, be the category of separable spaces which are, respectively, subfine, $\mathcal{M}$-fine, and measurable.

4.1. Theorem. *If $\mu X \in \mathscr{S}_i$, then $C(\mu X) \in \mathscr{A}_i(X)$. If $A \in \mathscr{A}_i(X)$, then there is unique $\mu X \in \mathscr{S}_i$ with $C(\mu X) = A$.*

For $i = 1$, this is, apart from the uniqueness, due to Isbell, Ginsberg, and Corson. The proofs of the first statements are fairly direct. For the existence in the second, we require three constructions: (cc) Embed $X \to P \equiv \prod\{R_f : f \in A\}$ in the usual way. Then A becomes $C(P)/X$, and $\mu X = \alpha P/X$. (ci) Essentially use 3.2 (b), in the form $C(\mu X) =$ uniform closure of $\{f/g : f,\ g \in A^*\}$. (regular) Prove that $\operatorname{coz} A$ is a σ-field, and A is the measurable functions. (In each case, μ turns out to be the coreflection of the A-weak uniformity.)

It suffices to have uniqueness for $i = 1$, and this is the theorem following. c is the functor (reflector) which assigns to μX the space $c\mu X$ generated by $C(\mu X)$.

4.2. Theorem. *If $\mu X \in \mathscr{S}_1$, then $lc\mu X = \mu X$. Hence, $(c/\mathscr{S}_1)^{-1} = l/c(\mathscr{S}_1)$, and $c/\mathscr{S}_1$ is an isomorphism of categories.*

The proof [4d] is quite difficult. (It is essentially the proof that $\mu X \in \mathscr{S}$ iff μX is $\{\varrho\mathscr{R}^{\aleph_0}\}$-fine and $\alpha\mathscr{R}^{\aleph_0}$-weak, of 2.2.) Not surprisingly, there are easy proofs, based on 3.1, of the weaker theorems that if $\mu X \in \mathscr{S}_2$ or $\mathscr{S}_3$ then $mc\mu X = \mu X$, or $bc\mu X = \mu X$. For $i = 2$, even the following is easy. p denotes the reflector into precompact spaces: $p\mu X$ has basis finite μ-covers (and is generated by $C^*(\mu X)$).

4.3. Theorem. *If $\mu X \in \mathscr{S}_2$, then $mp\mu X = \mu X$. Hence, $(p/\mathscr{S}_2)^{-1} = m/p(\mathscr{S}_2)$, and $p/\mathscr{S}_2$ is an isomorphism of categories.*

Both 4.1 and 4.2 are sharp: $p/\mathscr{S}_1$ is not one-to-one; c is not one-to-one on the category of separable $\{\varrho\mathscr{R}^{\aleph_0}\}$-fine spaces, which is only "slightly larger" than $\mathscr{S}_1$. But 4.1 and 4.2 hardly exhaust the subject: Smirnov has shown that p is one-to-one on metric spaces (hence so is c).

Consider Archimedean lattice-ordered algebras over R, with identity a weak order unit, the Φ-algebras of Henrikson-Johnson. If A is one, let $\mathscr{R}(A)$ be the set of real ideals of A, those M with $A/M = R$. If $\bigcap\mathscr{R}(A) = \{0\}$, then $A \ni a \to \bar{a}: \mathscr{R}(A) \to$ $\to R$ defined by $\bar{a}(M) = a + M$, defines an isomorphism of A onto the function algebra $\bar{A}$: $\mathscr{R}(A) \to R$. Henrikson, Johnson, and Isbell give algebraic definitions of cc, ci, and regular such that when $\bigcap\mathscr{R}(A) = \{0\}$, A has the property iff $\bar{A}$ does (in the sense already used). Let $\mathscr{A}_i$, $i = 1, 2, 3$, be the category of Φ-algebras A with $\bigcap\mathscr{R}(A) = \{0\}$, which are, respectively, cc, ci, and regular (with Φ-algebra homomorphisms; equivalently, ring homomorphisms preserving 1). Let $\gamma\mathscr{S}_i$ be the category of complete $\mathscr{S}_i$-spaces.

4.4. Theorem. *$\gamma\mathscr{S}_i$ and $\mathscr{A}_i$ are dual.*

This follows from 4.2 and 4.1, upon identifying X, when $\mu X \in \gamma\mathscr{S}_i$, with the "real ideal space" of $C(\mu X)$. This identification results from the following. Let A: $X \to R$ be a point-separating ring and lattice, with $1 \in A$. Let μ_A be the weak uniformity on X generated by A.

4.5. Proposition. *$\mu_A X$ is complete iff each real ideal in A is of the form $\{f \in A : f(x) = 0\}$ for some $x \in X$.*

4.6. Remarks. Let $\mathscr{S}$ be the category of separable spaces. The coreflectors l, m, b restricted to $\gamma c(\mathscr{S})$ induce by duality reflectors of some category of vector lattices, say $\mathscr{L}$, onto $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$, respectively. $\mathscr{L}$ has not been indentified algebraically, though work of Fenstad and Császár is relevant; and the reflectors have not been studied carefully. But the coreflectors l, m, b, restricted to those objects of $\gamma c(\mathscr{S})$ associated with rings $C(\mu X)$ induce by duality reflectors of a category of Φ-algebras (which is not identified), and the reflectors have extensions to reflectors of all Φ-algebras (onto super-categories of the $\mathscr{A}_i$); these have been studied by Eleanor Aron. The category of uniform spaces for which $C(\mu X)$ is a ring is coreflective (Rice), and a construction of the coreflection exists (G. D. Reynolds); the category remains poorly understood.

See [5a] for a careful discussion of duality.

Added July 5, 1972. Isbell points out that 4.2. above follows from some results in [3], and that my work on $\mathscr{M}$-fine spaces overlaps with work of A. D. Alexandroff, Additive set functions in abstract spaces, Mat. Sb. *50* (1940), 307–348, *51* (1941), 563–628, *55* (1943), 169–238. The latter point is discussed in (revised) [4b].

There is a theorem similar to 4.2 and 4.3 implicit in [4c], concerning the category $\mathscr{S}_3$ and the Samuel compactification $s : s/\gamma\mathscr{S}_3$ *is an isomorphism of categories* (though $bs\mu X = \mu X$ rarely holds).

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