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Nice spaces; nice maps

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NICE SPACES; NICE MAPS

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1. Introduction

At the first Prague Symposium, P. S. Alexandroff [1] asked the question

"*Which topological spaces can be characterized as 'nice' continuous images of 'nice' spaces?*"

Much work has been done on this topic in ten years; this paper reviews some of this work.

In this paper all maps are considered to be both continuous and onto; N denotes the natural numbers throughout.

2. Metric Spaces

Perhaps the first characterization theorem of this type was that done independently by Ponomarev and Hanai about 1961.

Theorem 2.1. *A T_0-topological space Y is first countable iff it is the image of a metric space X by an open map $f : X \to Y$.*

In 1963 A. Arhangelskii introduced the pseudo-open maps.

Definition 2.2. A map $f : X \to Y$ is *pseudo-open* iff for every $y \in Y$ and every neighborhood U of $f^{-1}(y)$, $y \in \operatorname{Int} f(U)$.

Pseudo-open maps are precisely those quotient maps which are hereditary. Arhangelskii then stated the next result.

Theorem 2.3. *Y is the pseudo-open image of a metric space X iff Y is a Fréchet space* (i.e., a set $F \subseteq Y$ is closed iff for every $p \in F$ there exists a point-sequence $\{p_n\} \subseteq F$ such that $\{p_n\} \to p$).

In 1965, S. P. Franklin tackled the problem of quotients of metric spaces, and in an interesting paper [6] showed the following.

Theorem 2.4. *Y is a sequential space iff Y is the image of a metric space under a quotient map.*

Sequential spaces are defined as follows.

Definition 2.5. Y is *sequential* iff: U is open in Y iff every sequence converging to a point in U is residually (eventually) in U.

In his paper Franklin also gives an explicit proof of Arhangelskii's assertion of Theorem 2.3.

As well as not being hereditary, quotient maps also do not preserve products. Thus it is logical to consider a map which is stronger than quotient but preserves products. This was done by E. Michael [13].

Definition 2.6. $f : X \to Y$ is a *bi-quotient map* iff: given $\mathscr{B}$ a filterbase in Y, if $y \in \mathrm{Cl}\, B$ for all $B \in \mathscr{B}$, then there exists $x \in f^{-1}(y)$ such that $x \in \mathrm{Cl}\, f^{-1}(B)$ for all $f^{-1}(B) \in f^{-1}(\mathscr{B})$.

Michael has asserted the next statement.

Theorem 2.7. *Y is the image of a metric space X by a bi-quotient map f iff every maximal filter $\mathscr{F} \to x$ has a countable family $\{F_1, F_2, \dots\} \subseteq \mathscr{F}$ such that $\{F_i\} \to x$.* (Such a space is called bi-sequential.)

P. Vopěnka has defined almost-open maps.

Definition 2.8. $f : X \to Y$ is *almost-open* iff for any $y \in Y$ there exists an $x \in f^{-1}(y)$ having a basis of open sets such that the image of each member of the basis is open.

Almost-open maps form a class between open and bi-quotient maps. The next statement is a simple modification of Ponomarev's Theorem 2.1.

Corollary 2.9. *Y is first countable iff Y is the image of a metric space under an almost-open map.*

F. Siwiec and V. Mancuso [26] altered the definition of bi-quotient maps as follows.

Definition 2.10. A map $f : X \to Y$ is *countably bi-quotient* iff, when $\{A_n : n \in N\}$ is a decreasing family accumulating at some $y \in Y$, then there exists an $x \in f^{-1}(y)$ such that $f^{-1}(A_n)$ accumulates at x.

Definition 2.11. Y is *countably bi-sequential* iff, when $\{F_n\}$ is a decreasing sequence of sets accumulating at $y \in Y$, there exist $y_n \in F_n$ such that the point-sequence $\{y_n\} \to y$.

Theorem 2.12. *Y is countably bi-sequential iff Y is the countably bi-quotient image of a metric space X.*

Perfect maps have long been known to preserve metrizability (see Dugundji [5]), as have simultaneously closed and open maps (sometimes called clopen). For a proof of this last, see Morita-Hanai [16].

Closed images of metric spaces have been characterized by N. Lašnev [11]. Their characterization is somewhat different from those discussed previously. First, let us state a definition.

Definition 2.13. A sequence of closed coverings $\mathscr{U}_i$ of a space X is said to be an *almost refining sequence* if and only if for any point x in X, any arbitrary system of sets $\{B_i\}$ such that $B_i \in \mathscr{U}_i$ and $x \in B_i$, is either hereditarily closure preserving or forms a network of the space at the point x.

Lašnev's result follows.

Theorem 2.14. *A T_1-space Y is the closed image of a metric space X iff*

a) *Y is Fréchet,*

b) *there exists in Y an almost refining sequence of hereditarily closure preserving closed covers forming a network.*

Such a space has been called a Lašnev space.

A useful map between closed and perfect is quasi-perfect.

Definition 2.15. $f: X \to Y$ is *quasi-perfect* iff f is closed and $f^{-1}(y)$ is countably compact for every $y \in Y$.

Quasi-perfect images of metric spaces are well known to be metric.

Definition 2.16. A map $f: X \to Y$ is an *s-map* iff the preimage of each point has a countable dense subset.

T. Hoshina [8] has characterized quotient s-images of metric spaces.

Definition 2.17. A T_1-space Y is a *q-s space* iff it has a point countable family A of subsets of X satisfying the following condition:

A set $F \subseteq Y$ is closed iff for every $x \in Y - F$ and each q-sequence $\{A_n : n \in N\} \subseteq A$ with $x = \bigcap\{A_n : n \in N\}$, $F \cap A_n = \emptyset$ for some $n \in N$.

Theorem 2.18. *A T_1-space Y is the quotient s-image of a metric space X iff Y is a q-s space.*

Ponomarev [22] has studied the problem of open s-images of metric spaces.

Theorem 2.19. *Y has a point countable base iff Y is the open s-image of a metric space X.*

The following tables collect our results thus far, assuming T_1.

$$\begin{tikzcd}
 & & \text{open } s\text{-map} \arrow[r] & \text{quotient } s\text{-map} \arrow[r] & \text{quotient} \\
\text{open} \arrow[r] & \text{almost-open} \arrow[r] & \text{bi-quotient} \arrow[r] & \text{countably bi-quotient} \arrow[r] & \text{pseudo-open} \arrow[u] \\
 & & \text{perfect} \arrow[r] & \text{quasi-perfect} \arrow[r] & \text{closed} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \text{point-countable base} \arrow[r] & q\text{-}s \arrow[r] & \text{sequential} \\
\text{metric} \arrow[r] & \text{first-countable} \arrow[r] & \text{bi-sequential} \arrow[r] & \text{countably bi-sequential} \arrow[r] & \text{Fréchet} \arrow[u] \\
 & & & & \text{Lašnev} \arrow[u]
\end{tikzcd}$$

3. *M*-Spaces.

In 1969, J. Nagata suggested extending the characterization problem to M-spaces, a class of spaces defined by K. Morita.

Definition 3.1. X is an M-space iff it has a normal sequence of open covers $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$ such that each point-sequence $\{x_n\}$, of the form $x_n \in \mathrm{St}(x, \mathcal{U}_n)$ for every $n \in N$ and for fixed $x \in X$, has a cluster point.

M-spaces simultaneously generalize all metric spaces and all countably compact spaces. These spaces have been characterized as follows by Morita [17].

Theorem 3.2. *X is M iff it is the quasi-perfect inverse image of a metric space.*

It is thus clear that quasi-perfect images of M-spaces are M.

M-spaces are not necessarily paracompact, so the problem of characterization may also be studied for paracompact M-spaces. In the next tables and definitions we see the results of this study. (For proofs, see [3], [18], [20], [30], [23] and [27].)

Table 3.3. ***M*-spaces**

f	Y	additional condition
(1) open	q	Y regular
(2) almost-open	q	Y regular
(3) bi-quotient	bi-quasi-k	
(4) countably bi-quotient	countably bi-quasi-k	
(5) pseudo-open	singly bi-quasi-k	
(6) quotient	quasi-k	X, Y regular
(7) perfect	M	Y normal
(8) closed and open	M	X normal

Definition 3.4. A countable decreasing family of sets $\{U_1, U_2, \ldots\}$ is called a *q-sequence at* $y \in Y$ iff $y \in U_n$ for every $n \in N$ and if $\{y_n\}$ is a point-sequence such that $y_n \in U_n$ for every $n \in N$, then $\{y_n\}$ clusters.

Definition 3.5. Y is a *q-space* iff every $y \in Y$ has a q-sequence of neighborhoods.

Definition 3.6. Y is *bi-quasi-k* iff every maximal filter $\mathscr{F} \to y \in Y$ contains a subfilter $\{F_1, F_2, \ldots\}$ which is a q-sequence at y.

Definition 3.7. Y is *countably bi-quasi-k* iff for every decreasing family $\{F_n\}$ clustering at $y \in Y$, there exists a q-sequence $\{A_n\}$ at y such that $y \in \mathrm{Cl}(F_n \cap A_n)$ for every $n \in N$.

Definition 3.8. Y is *singly bi-quasi-k* iff the following holds: given $B \subseteq Y$, $y \in \mathrm{Cl}\, B$ iff there exists a q-sequence $\{U_1, U_2, \ldots\}$ at y such that $y \in \mathrm{Cl}(B \cap U_n)$ for every $n \in N$.

Definition 3.9. Y is *quasi-k* iff the following holds: $F \subseteq Y$ is closed in Y iff $F \cap C$ is relatively closed in C for every countably compact $C \subseteq Y$.

Table 3.10. Paracompact M-spaces

f	Y	additional conditions
(1) open	pointwise countable type	Y Hausdorff
(2) almost-open	pointwise countable type	Y Hausdorff
(3) bi-quotient	bi-k	
(4) countably bi-quotient	countably bi-k	
(5) pseudo-open	singly bi-k	
(6) quotient	k	X, Y regular
(7) perfect	paracompact M	Y normal
(8) closed and open	paracompact M	X, Y normal

Definition 3.11. Given a point $y \in Y$, a countable decreasing family $\{U_1, U_2, \ldots\}$ is said to form a *k-sequence* at y iff $\{U_1, U_2, \ldots\}$ is a q-sequence at y and $\bigcap\{U_n : n \in N\}$ is compact.

Definitions of the spaces given in Table 3.10 will be omitted, except to say that they are the same as those in Table 3.3 with the exceptions that the words "k-sequence" and "compact" replace the words "q-sequence" and "countably compact", respectively.

Similarly, $\mathfrak{C}$-spaces were defined by Ishii, Tsuda and Kunugi [10]. Such spaces are defined as follows.

Definition 3.12. A topological space X is said to be a *$\mathfrak{C}$-space* iff X has a normal sequence of open covers $\mathscr{U}_1, \mathscr{U}_2, \ldots$ such that every point-sequence $\{x_n\}$, of the form $x_n \in \mathrm{St}(x, \mathscr{U}_n)$ for every $n \in N$ and for fixed $x \in X$, has a subsequence with compact closure.

These spaces include paracompact M-spaces, and form a class of spaces whose product with every M-space is M.

Definition 3.13. Given $y \in Y$, a countable decreasing family of sets $\{U_1, U_2, \ldots\}$ is said to form an *r_0-sequence at y* iff it is a q-sequence at y and every point-sequence $\{x_n\}$, $x_n \in U_n$ for every $n \in N$, has a subsequence with compact closure.

Definition 3.14. A set $P \subseteq Y$, a topological space, is said to be *proto-compact* iff every sequence of points of P has a subsequence with compact closure.

Relative to these last two definitions, we can now form another table. Again definitions of the various spaces are omitted for brevity. (For a full description of results, see [10] and [25].)

Table 3.15. $\mathfrak{C}$-spaces

f	Y	additional conditions
(1) almost-open	r_0	Y regular
(2) bi-quotient	bi-proto-k	
(3) pseudo-open	singly bi-proto-k	
(4) quotient	proto-k	X, Y regular
(5) perfect	$\mathfrak{C}$	Y normal

4. Locally compact and separable metric spaces

Locally compact and separable metric spaces have been characterized by means of mappings. The next result, due to K. Morita, is well-known.

Theorem 4.1. *Local compactness is invariant under open maps.*

Michael has also shown [13] that bi-quotient maps preserve local compactness. Another result, due to D. E. Cohen [4], is by now classical.

Theorem 4.2. *Y is a k-space iff it is the quotient of a locally compact space X.*

Bi-quotient and countably bi-quotient maps do not have the same effect on locally compact spaces. The next result is due to F. Siwiec [27] and Michael [14].

Theorem 4.3. *Y is strongly k' iff it is the image of a locally compact space under a countably bi-quotient map.* (A *strongly k'-space* Y is one in which, given a decreasing sequence $\{U_n\}$ accumulating at $y \in Y$, there exists a compact $K \subseteq Y$ such that $y \in \mathrm{Cl}(U_n \cap K)$ for all $n \in N$.)

Images of locally compact spaces by pseudo-open maps are precisely the k'-spaces. This result is due to Arhangelskii [2].

Definition 4.4. Given a set $A \subseteq Y$, topological, suppose a point $y \in \mathrm{Cl}\, A$ iff there exists a compact set $K \subseteq Y$ such that $y \in \mathrm{Cl}\,(K \cap A)$. Then Y is said to be a *k'-space.*

E. Michael [12] has discussed two classes of spaces which relate to separable metric spaces.

Definition 4.5. An *$\aleph_0$-space* Y is a regular space with a countable k-network (a *k-network* is a family $\mathscr{P}$ of subsets P of Y such that if C is compact, U open in Y and $C \subseteq U$, then some $P \in \mathscr{P}$ has the property $C \subseteq P \subseteq U$).

Definition 4.6. A *cosmic space* Y is a regular space with a countable network.

Theorem 4.7. *Y is a cosmic space iff it is the continuous image of a separable metric space X.*

Michael has also shown, in the same paper, the next result.

Theorem 4.8. *Y is $\aleph_0$ and k iff Y is the quotient of a separable metric space.*

P. Strong [28] has noted that Y need only have a countable k-network, be sequential and Hausdorff in the above result. He has also shown the next statement.

Theorem 4.9. *Y is a Fréchet, Hausdorff space with a countable k-network iff Y is the image of a separable metric space by a pseudo-open map.*

Separable metric spaces are preserved under open, bi-quotient and countably bi-quotient maps (whenever the image space is regular).

5. Other results.

Many maps and spaces have occurred but once in characterization theorems. Hence results of this section will not seem as fully developed as in previous discussions. Questions will naturally occur to the reader.

Definition 5.1. $f : X \to Y$ is *compact-covering* if every compact set $K \subseteq Y$ is the image of a compact $C \subseteq X$.

Michael defined the above map and proved the next result [12].

Theorem 5.2. *Y is $\aleph_0$ iff Y is the image by a compact-covering map of a separable metric space.*

Definition 5.3. Y is a *c-space* iff the closure of any set U in Y is obtained from closures of countable sets in U.

Theorem 5.4. *Y is a c-space iff Y is the quotient image of copies of a countable discrete space with single points of its Stone-Čech compactification.*

The above characterization is due to this author [24], but such spaces were originally mentioned by S. G. Mrówka and R. C. Moore [15] under the title of "spaces determined by countable sets".

Nagata has recently shown [21] that images of T_1 M-spaces under perfect maps are precisely the M^*-spaces of T. Ishii.

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