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SOME TIETZE TYPE EXTENSION THEOREMS

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In the historical development of the separation axioms in set-theoretical topology, the normal topological spaces received appropriate recognition for their capability to determine, set-theoretically, the existence of non-trivial continuous real valued functions. In fact they are precisely the class of topological spaces in which every closed subset is C-embedded (or C^*-embedded). This is recorded for us in the famous Tietze-Urysohn Extension Theorem.

Since the study of continuous real valued functions was at the core of the early developments of topology, this class of spaces satisfied, then, the desires of many. On the other hand, however, normal spaces do not yield to many of the topological operations we often like to perform on classes of spaces. For example, the category of normal topological spaces is neither closed under finite products nor hereditary for arbitrary subspaces. The standard examples of the Sorgenfrey plane and the Tychonoff plank are appropriate here.

Thus the question as to when the product set of two normal spaces is again normal in its Tychonoff product topology has been of some interest. In particular, in homotopy theory one often likes to know when the product space, $X \times I$, is normal where I is the closed unit interval of real numbers and where X is some normal topological space. In dimension theory the normality of the product $X \times Y$ often appears in the dimension product theorem for any of the various concepts of dimension that occur in spaces which are not separable metric spaces.

The normality of $X \times I$ was settled by C. H. Dowker in [6]. He showed that for any compact metric space A, $X \times A$ is a normal (respectively, collectionwise normal) Hausdorff space if and only if X is a countably paracompact, normal, (respectively, collectionwise normal) Hausdorff space. Also M. E. Rudin in [11] has shown the existence of a collectionwise normal (and therefore normal) Hausdorff space which is not countably paracompact. H. Tamano in [14] has shown that $X \times A$ is normal Hausdorff for any compact Hausdorff space A if and only if X is a paracompact Hausdorff space. K. Morita in [10] has demonstrated that $X \times A$ is normal for any metric space A if and only if X is a normal P-space. Not so easy to describe are the conditions on X, as recently discovered by Y. Katuta in [9], for the equivalent formulation of $X \times A$ being normal when A is a paracompact Hausdorff space.

In another direction the strengthening of normality to collectionwise normality in [5] by R. H. Bing assisted in resolving the metrization problem. It has also been shown to be useful in extension theory. In fact, a space X is collectionwise normal if and only if every closed subset is P-embedded (that is, every continuous pseudometric on the subset extends to a continuous pseudometric on the entire space) in the space (see [7] and [12]).

The concept of P-embedding is definitely stronger than that of C-embedding which in turn is used to characterize the class of normal spaces. Also every paracompact Hausdorff space is collectionwise normal. Consequently compact subsets of Tychonoff spaces are P-embedded.

In [4], R. Arens began the serious consideration of P-embedding. Some characterizations of it were given, reminiscent of the Tietze-Urysohn extension theorem for normal spaces. In [1], L. I. Sennott and the author showed the following.

Theorem 1. *Let S be a subset of a non-empty topological space X. Then S is P-embedded in X if and only if every continuous function from S into a bounded, closed, convex subset of a Banach space extends to a continuous function on X.*

Thus, as a corollary, collectionwise normal spaces can be characterized in the sense of Tietze-Urysohn as shown in (2) of the following corollary.

Corollary 1.1. *For a non-empty topological space X, the following statements are equivalent*:

(1) *The space X is collectionwise normal.*

(2) *For every closed subset F of X, every continuous function from F into a bounded closed convex subset B of a Banach space can be extended continuously to X (and the range of the extension is contained in B).*

(3) *For every closed subspace F of X, the product space $F \times X$ is C^*-embedded in $F \times \beta X$, where βX is the Čech-Stone extension of X.*

The third statement in this corollary is due to H. Tamano in [15].

In [3], Arens asked whether a continuous function from a closed subset of a normal space into a bounded closed convex subset C of a Banach space could be extended continuously to the whole space with values still in the subset C. But in [5], Bing gave an example of a normal space which is not collectionwise normal. Thus the corollary gives a negative reply to Arens' original query.

Looking at Corollary 1.1, one asks if a similar statement may be made regarding normal spaces. Such was already done in [3]. However in [1], the authors were able to give a stronger version of this by first proving the following formulation of C-embedding.

Theorem 2. *The following statements are equivalent for a non-empty subspace S of a topological space X.*

(1) *The subspace S is C-embedded (respectively C^*-embedded) in X.*

(2) *Given a complete convex metrizable subset M of a locally convex topological vector space L, every continuous function f on S with $f(S)$ contained in M and separable (respectively, totally bounded) has a continuous extension f^* to X with $f^*(X)$ contained in M.*

(3) *Every continuous function f from S to a Fréchet space (that is, a complete, metrizable, locally convex topological vector space) such that $f(S)$ is separable (respectively, totally bounded) extends to a continuous function on X.*

Thus the sharpened version of the result in [3], reads as

Corollary 2.1. *A non-empty topological space X is normal if and only if for every closed subset F of X, and for any continuous function f from F into a closed, convex metrizable subset M of a locally convex topological vector space L such that $f(F)$ is contained in M and is separable, there is a continuous extension f^* of f to X where $f^*(X)$ is contained in M.*

The third statement in Theorem 2 says that

Corollary 2.2. *The Hewitt realcompactification υX of a Tychonoff space X is that unique realcompactification of X for which every continuous function f from X into a Fréchet space, such that $f(X)$ is separable, can be extended to a continuous function on υX.*

In [12], it was shown that if the cardinality of a dense C-embedded subset S of a Tychonoff space X is non-measurable, then S is P-embedded in X. From this 2.2 can be restated for Tychonoff spaces of non-measurable cardinality by dropping the requirement that $f(X)$ be separable.

As Corollary 2.2 characterized the Hewitt realcompactification, the Čech-Stone compactification may be characterized by considering the equivalent formulations for C^*-embedding in Theorem 2.

Corollary 2.3. *The Čech-Stone compactification of a Tychonoff space X is that unique compactification βX of X for which every continuous function f from X to a Fréchet space, such that $f(X)$ is totally bounded, can be extended to a function on βX.*

Let us point out here that in the proof of statement (1) implies that of (2) in Theorem 2, one may adapt the proof to show the following (see [1] for details) which is an improvement of a result in [8].

Theorem 3. *If S is a non empty subspace of a uniform space X and if L is any Fréchet space, then every uniformly continuous function from S into L can be extended to a continuous function on X.*

Let us say that a subset S is *strongly P-embedded* (respectively, *strongly C-embedded*) in the space X if for every σ-locally finite (respectively, every countable) open cover $\mathscr{U}$ of S there is a locally finite cozero set cover $\mathscr{V}$ of X such that $\mathscr{V} \mid S = \{V \cap S : V \in \mathscr{V}\}$ refines $\mathscr{U}$.

In [13], it was shown that a subset S is P-embedded in X if and only if every σ-locally finite cozero set cover $\mathscr{U}$ of S has a locally finite cozero set cover $\mathscr{V}$ of X such that $\mathscr{V} \mid S$ refines $\mathscr{U}$. Consequently strongly P-embedding is definitely stronger than P-embedding.

In [2], we have shown that a strongly P-embedded subset is strongly C-embedded and that a strongly C-embedded subset is C-embedded. Also none of the implications are reversible.

Countably paracompact spaces have been mentioned above regarding the question of normality of the product. Also P-embedding, C-embedding and C^*-embedding have been considered. Let us now put these together with countable paracompactness.

Theorem 4. *For T_1 spaces X, the following statements are equivalent.*

(1) *The space X is collectionwise normal and countably paracompact.*
(2) *Every closed subset F of X is strongly P-embedded in X.*
(3) *The product set $X \times I$ is collectionwise normal.*

Corollary 4.1. *Compact subsets of Tychonoff spaces are strongly P-embedded.*

Statement (2) of Theorem 4 is shown in [2] as is statement (2) of the following.

Theorem 5. *For T_1 spaces X, the following statements are equivalent.*

(1) *The space X is normal and countably paracompact.*
(2) *Every closed subset F is strongly C-embedded in X.*
(3) *The product set $X \times I$ is normal.*

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