ON SOME PRETOPOLOGIES ASSOCIATED WITH A TOPOLOGY

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Given a set X and a family $\mathcal{U}$ of subsets of X, let $(X, \mathcal{U})$ be a *pretopological space* if $\mathcal{U}$ contains X and the empty set and is closed under arbitrary union. Various notions like the interior, closure, frontier, and the derived set operators as well as set properties like open, closed, dense, nowhere dense, dense-in-itself, connected, compact etc. can be defined in a pretopological space in analogy with a topological space. Many results of topological spaces remain valid in pretopological spaces, whereas some become false.

Let, for every set $A \subset X$, $A^\alpha = A^{-0}$, $A^\beta = A^{0-}$, $A^\gamma = A^{0-0}$, $A^\delta = A^{-0-}$, $A^\xi = A^\alpha \cap A^\beta$ and $A^\eta = A^\alpha \cup A^\beta$. ($A^-$ means the closure of A, A^0 the interior of A.) Let $\mathcal{U}_\lambda = \{A \subset X : A \subset A^\lambda\}$ for $\lambda = \alpha, \beta, \gamma, \delta, \xi$ and η. Each of these families of sets is again a pretopology on X and we have, in general,

$$\mathcal{U} \subset \mathcal{U}_\gamma = \mathcal{U}_\xi \subset \mathcal{U}_\alpha[\mathcal{U}_\beta] \subset \mathcal{U}_\eta \subset \mathcal{U}_\delta .$$

We call these pretopologies the *associated pretopologies* of $\mathcal{U}$. If X has no isolated points then $\mathcal{U}_d = \{A \subset X : A \subset A^d\}$, where A^d denotes the derived set of A, is also a pretopology on X. In case $\mathcal{U}$ is a topology, it is only $\mathcal{U}_\gamma$ which is always a topology and the rest are in general only pretopologies. This is why we investigate pretopological spaces.

Let a pretopological space $(X, \mathcal{U})$ be a *pre*topological* space if $(A \cap A^\alpha)^\alpha = A^\alpha$ for every set $A \subset X$. A topological space is pre*topological and all the pretopologies associated with a pre*topology are pre*topologies. The topological structure of the pretopologies associated with a pre*topology are investigated and some of the properties of the original space are determined in terms of these asociated pretopologies. The pretopologies associated with the associated pretopologies of a pre*-topology are obtained as one of the original associated pretopologies.

The frontier of a set can be decomposed into six parts in such a way that various properties of sets and spaces are characterized in terms of these parts. The decomposed frontiers relative to the associated pretopologies are obtained in terms of the original pretopology.

The notion of continuity has a natural extension to functions in pretopological spaces. Many results on continuity in topological spaces remain valid in pretopological spaces.

Given $f : X \to Y$, where $(X, \mathcal{U})$, $(Y, \mathcal{V})$ are pretopological spaces, let, for each of $\lambda = \alpha, \beta, \gamma, \delta, \xi, \eta$, f be *λ-weakly continuous* if $f^{-1}(V) \in \mathcal{U}_\lambda$ for every $V \in \mathcal{V}$. In topological spaces the quasi-continuity and near continuity are characterized by β-weak continuity and α-weak continuity, respectively. It follows from the established results that these two notions of continuity yield together θ-continuity, which in turn implies continuity in case Y is regular. Whereas it is known that dense sets are the stationary sets for quasi-continuity, it turns out that the stationary sets for near continuity are the sets with nowhere dense complements.

We further investigate continuity and openness of the homomorphisms between topological groups in terms of much weaker properties.