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In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 405--410.

Persistent URL: http://dml.cz/dmlcz/700757

PROPERTIES OF EXPANDABLE SPACES

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Introduction. In [7] J. C. Smith and L. L. Krajewski investigated the properties of spaces in which locally finite collections can be expanded to point-finite or locally finite open collections. A number of results concerning these "expandable" spaces were obtained as well as the relationship of this class of spaces with the class of collectionwise normal spaces. In this paper we investigate the properties of the class of spaces in which locally finite collections can be expanded to point-finite or locally finite cozero collections. In § 1 results analogous to those in [7] are obtained for "cozero-expandable" spaces. In § 2 it is shown that every expandable screenable space is paracompact. In § 3 we observe that a class of spaces weaker than the class of collectionwise normal spaces has properties similar to the class of collectionwise normal spaces.

1. Definitions and preliminary results

Definition 1.1. A space X is called *cz-expandable* if for every locally finite collection $\{F_\alpha : \alpha \in A\}$ of subsets of X, there exists a locally finite collection $\{G_\alpha : \alpha \in A\}$ of cozero subsets of X, such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in A$. We will refer to collections with the above property as being "cozero expandable".

Remark. The above definition is analogous to the definitions of various "expandable" spaces found in [7]. By restricting the cardinality of the index set A, analogous cardinality-dependent definitions can be obtained. These definitions are omitted. The reader is referred to [7] for the definitions of expandable spaces, almost expandable spaces, discretely expandable spaces, etc.

Definition 1.2. (i) A space X is called (*almost*) *cz-expandable* if every locally finite collection of subsets of X can be expanded to a (point-finite) locally finite cozero collection.

(ii) A space X is called (*almost*) *discretely cz-expandable* if every discrete collection of subsets of X can be expanded to a (point-finite) locally finite cozero collection.

(iii) A space X is called (*almost*) *boundedly cz-expandable* if every bounded locally finite collection of subsets of X can be expanded to a (point-finite) locally finite cozero collection.

The following two theorems are obvious from the results in [7] and the fact that the locally finite union of cozero sets is a cozero set.

Theorem 1.3. *Let X be a normal space. Then*

(i) *X is (almost) expandable if and only if X is (almost) cz-expandable.*

(ii) *X is (almost) discretely expandable if and only if X is (almost) discretely cz-expandable.*

(iii) *X is (almost) boundedly expandable if and only if X is (almost) boundedly cz-expandable.*

Theorem 1.4. (i) *A space X is almost discretely expandable if and only if X is almost boundedly expandable.*

(ii) *A space X is (almost) discretely cz-expandable if and only if X is (almost) boundedly cz-expandable.*

Theorem 1.5. (Morita and Dowker). *Let $\{G_1, G_2, \ldots\}$ be a countable collection of cozero subsets of X and $G = \bigcup_{i=1}^{\infty} G_i$. Then there exists a collection of cozero subsets $\{H_1, H_2, \ldots\}$ such that $G = \bigcup_{i=1}^{\infty} H_i$ and,*

(i) *$H_i \subseteq G_i$ for each i,*

(ii) *$\{H_1, H_2, \ldots\}$ is locally finite with respect to points in G. That is, for each $x \in G$, there exists an open (in X) neighborhood of x which intersects only finitely many members of $\{H_1, H_2, \ldots\}$.*

Proof. The proof is found in Theorem 2.7 of [5].

We now obtain a result analogous to Theorem 2.8 of [7].

Theorem 1.6. *A space X is cz-expandable if and only if X is discretely cz-expandable and $\aleph_0$-cz-expandable.*

Proof. The sufficiency is clear.

Necessity. Let $\mathscr{F} = \{F_\alpha : \alpha \in A\}$ be a locally finite collection of closed subsets of X. For each integer $n \geqq 0$, define $S_n = \{x \in X : \operatorname{ord}(x, \mathscr{F}) \leqq n\}$ so that $\{S_0, S_1, \ldots\}$ is an open cover of X. Since X is countably paracompact and $S_n \subseteq S_{n+1}$ for each n, there exists by Theorem 5 of [2] a locally finite open cover $\{U_0, U_1, \ldots\}$ of X, such that $U_n \subseteq \overline{U}_n \subseteq S_n$. Now define $\mathscr{F}_n = \{\overline{U}_n \cap F_\alpha : \alpha \in A\} = \{F(n, \alpha) : : \alpha \in A\}$ for each n, so that $\mathscr{F}_n$ is a bounded locally finite collection in X. By Theorem

1.4 above there exists for each n, a locally finite cozero collection $\mathcal{G}_n = \{G(n, \alpha) : : \alpha \in A\}$ such that $F(n, \alpha) \subseteq G(n, \alpha)$ for each $\alpha \in A$. Furthermore, since X is $\aleph_0$-*cz*-expandable, there exists a locally finite cozero collection $\{V_0, V_1, \ldots\}$ such that $\bar{U}_n \subseteq V_n$. Now define $G^*(n, \alpha) = G(n, \alpha) \cap V_n$ and let $G_\alpha^* = \bigcup_{n=0}^{\infty} G^*(n, \alpha)$ for each $\alpha \in A$. Then $\{G_\alpha^* : \alpha \in A\}$ is a locally finite cozero collection such that $F_\alpha \subseteq G_\alpha^*$ for each $\alpha \in A$. Hence X is *cz*-expandable.

2. Subset theorems and screenable spaces

Theorem 2.1. *The Closed Subset Theorem holds for the following properties:*

(i) *cz-expandable,*
(ii) *almost cz-expandable,*
(iii) *discretely cz-expandable,*
(iv) *almost discretely cz-expandable.*

Proof. The proofs follow in the same manner as those in Theorem 7.2 of [7].

In [7] the authors left the following question open: Are expandable screenable spaces paracompact? We now answer this question in the affirmative.

Theorem 2.2. *Every expandable screenable space is paracompact.*

Proof. Let $\mathcal{G}$ be an open cover of X. Then $\mathcal{G}$ has a σ-disjoint open refinement $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$, where $\mathcal{U}_i = \{U_\alpha : \alpha \in A_i\}$. Define $U_i = \bigcup_{\alpha \in A_i} U_\alpha$ so that $\{U_1, U_2, \ldots\}$ is a countable open cover of X. Since X is countably paracompact, there exists a locally finite open refinement $\{V_1, V_2, \ldots\}$. Now let $\mathcal{V}_i = \{V_i \cap U_\alpha : \alpha \in A_i\}$ and $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$. Then $\mathcal{V}$ is a point-finite open refinement of $\mathcal{G}$ so that X is metacompact. As consequence of Theorem 4.2 of [7], X is paracompact.

Remark. It is well known that locally finite cozero covers are normal. In fact if X is normal, then every countable point-finite open cover is normal. The following result illustrates a somewhat unusual property enjoyed by discretely *cz*-expandable spaces.

Definition 2.3. A cover $\mathcal{G}$ of a space X is called a *cz-θ-refinable cover* if $\mathcal{G}$ has a refinement $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$ satisfying:

(i) $\mathcal{U}_i$ is a cozero cover of X.

(ii) For $x \in X$ there exists an integer $n(x)$ such that x belongs to only finitely many members of $\mathcal{U}_{n(x)}$.

Theorem 2.4. *Let X be discretely cz-expandable. Then every cz-θ-refinable cover of X is normal.*

Proof. Let $\mathscr{G}$ be an open cover of X and $\mathscr{U} = \bigcup_{i=1}^{\infty} \mathscr{U}_i$ be a refinement of $\mathscr{G}$ satisfying the above properties. For each i we construct a sequence $\{\mathscr{G}(i,j) : j = 0, 1, \ldots\}$ of open collections such that

(1) $\mathscr{G}(i,j)$ is a locally finite cozero collection for each j.

(2) Each member of $\mathscr{G}(i,j)$ is a subset of some member of $\mathscr{U}_i$.

(3) If $x \in X$ and x belongs to at most j members of $\mathscr{U}_i$, then $x \in \bigcup\{G \in \mathscr{G}(i,k) : : 0 \leqq k \leqq j\}$.

(4) If $x \in \bigcup\{G : G \in \mathscr{G}(i,j)\}$ then x belongs to at least j members of $\mathscr{U}_i$.

The proof is by induction on j. Define $\mathscr{G}(i,0) = \emptyset$ and assume $\mathscr{G}(i,j)$ has been constructed satisfying (1)–(4) above for $0 \leqq j \leqq n$. We now construct $\mathscr{G}(i, n+1)$.

Let $\mathscr{U}_i = \{U_\alpha : \alpha \in A_i\}$, $\mathscr{B} = \{B \subseteq A_i : |B| = n+1\}$ and $G(i,j) = \bigcup\{G : G \in \in \mathscr{G}(i,j)\}$. Define $F(B) = [X - \bigcup_{j=0}^{n} G(i,j)] \cap [X - \bigcup\{U_\alpha : \alpha \in A_i - B\}]$ for each $B \in \mathscr{B}$. Then $\mathscr{F} = \{F(B) : B \in \mathscr{B}\}$ is a closed collection. We assert that $\mathscr{F}$ is discrete. Let $x \in X$.

Case 1: x belongs to more than $n+1$ members of $\mathscr{U}_i$, say $U(i,1), U(i,2), \ldots$ $\ldots, U(i,k)$. Then $U(x) = \bigcap_{j=1}^{k} U(i,j)$ is an open neighborhood of x which intersects no member of $\mathscr{F}$.

Case 2: x belongs to less than $n+1$ members of $\mathscr{U}_i$. From (3) above $x \in \bigcup\{G \in \mathscr{G}(i,j) : 0 \leqq j \leqq n\}$ which intersects no member of $\mathscr{F}$.

Case 3: x belongs to exactly $n+1$ members of $\mathscr{U}_i$; say $\{U_{\alpha_k} : k = 1, 2, \ldots, n+1\}$ Then $U(x) = \bigcap_{k=1}^{n+1} U_{\alpha_k}$ intersects no member of $\mathscr{F}$ other than $F(B)$ where $B = \{\alpha_1, \alpha_2, \ldots \ldots, \alpha_{n+1}\}$.

Since X is discretely cz-expandable, there exists a locally finite cozero collection $\mathscr{K} = \{K(B) : B \in \mathscr{B}\}$ such that $F(B) \subseteq K(B)$ for each $B \in \mathscr{B}$. Now define $L(B) = K(B) \cap [\bigcap_{\alpha \in B} U_\alpha]$ and $\mathscr{G}(i, n+1) = \{L(B) : B \in \mathscr{B}\}$. It is clear that (1), (2), and (4) above are satisfied by $\mathscr{G}(i, n+1)$. To show (3), let $x \in X$ such that x belongs to at most $n+1$ members of $\mathscr{U}_i$. If $x \notin \bigcup_{j=0}^{\infty} G(i,j)$, then $x \in F(B)$ for some $B \in \mathscr{B}$ and hence belongs to some member of $\mathscr{G}(i, n+1)$. Now define $H(i,j) = \bigcup\{G : G \in \mathscr{G}(i,j)\}$ so that $\mathscr{H} = = \{H(i,j) : i = 1, 2, \ldots; j = 0, 1, \ldots\}$ is a countable cozero cover of X. By Theorem 1.5 above $\mathscr{H}$ has a locally finite cozero refinement $\mathscr{H}^* = \{H^*(i,j) : i = 1, 2, \ldots; j = = 0, 1, \ldots\}$ such that $H^*(i,j) \subseteq H(i,j)$. Let $\mathscr{G}^*(i,j) = \{G \cap H^*(i,j) : G \in \mathscr{G}(i,j)\}$

for each i and each j. Then $\mathcal{G}^* = \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{\infty} \mathcal{G}^*(i,j)$ is a locally finite cozero refinement of $\mathcal{G}$. Since $\mathcal{G}^*$ is normal, $\mathcal{G}$ is normal.

Corollary 2.5. *Let X be a discretely cz-expandable space. Then every point-finite cozero cover of X is normal.*

3. *PF*-normal spaces

Definition 3.1. A space X is called *PF-normal* if every point-finite open cover is normal.

Theorem 3.2. *Collectionwise normal $\Rightarrow$ PF-normal $\Rightarrow$ normal.*

Proof. The proof of the first implication is Theorem 2 of [3] and the proof of the second is obvious.

Remark. The following examples show that neither of the reverse implications above is true.

Example 3.3. The space F of Bing [1] is *PF*-normal but not collectionwise normal. Michael [3] has shown that this space has the property that every point-finite open cover has a locally finite open refinement and hence is normal.

Example 3.4. Michael's modification of Bing's example G is normal but not *PF*-normal. Otherwise G would be fully normal and hence collectionwise normal.

The above examples show that closed subsets of *PF*-normal spaces need not have this property. It should be noted however that *PF*-normal spaces have many of the properties enjoyed by collectionwise normal spaces.

Theorem 3.5. *Let X be a PF-normal space. Then*

(i) *X is paracompact if and only if X is metacompact.*
(ii) *X is expandable if and only if X is almost expandable.*
(iii) *X is discretely expandable if and only if X is almost discretely expandable.*

Proof. Part (i) is obvious. Part (iii) follows from the same argument as does part (ii). We now prove part (ii). Let $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ be a locally finite collection of closed subsets of X. Since X is almost expandable, there exists a point-finite collection $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ of open subsets of X such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in A$. Since $\mathcal{G}^* = \mathcal{G} \cup \{X - \bigcup_{\alpha \in A} F_\alpha\}$ is a point-finite open cover of X, $\mathcal{G}^*$ is normal. Hence there exists a locally finite open cover $\mathcal{V} = \{V_\delta : \delta \in D\}$ such that $\{\mathrm{St}(V_\delta, \mathcal{V}) : \delta \in D\}$

refines $\mathscr{G}^*$. Define $H_\alpha = \mathrm{St}(F_\alpha, \mathscr{V})$ for all $\alpha \in A$. Then $\mathscr{H} = \{H_\alpha : \alpha \in A\}$ is a locally finite open collection such that $F_\alpha \subseteq H_\alpha$ for all $\alpha \in A$. Therefore X is expandable.

Corollary 3.6. *A space X is PF-normal and countably paracompact if and only if every σ-point-finite open cover is normal.*

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