Toposym 3

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A generalization of perfect maps

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A GENERALIZATION OF PERFECT MAPS

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This is a preliminary version of work that was done in cooperation with S. P. Franklin and D. Pumplün. Details, applications, and examples will appear elsewhere.

A. Perfect Maps. Assume all spaces to be completely regular.

Proposition 1. *For each continuous function* $f : X \to Y$ *the following conditions are equivalent*:

(1) f *is perfect,*

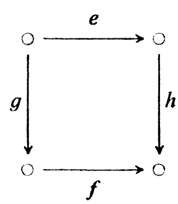
(2) *for each space* Z *the function* $f \times 1_Z$ *is perfect,*

(3) *each pullback of* f *is perfect,*

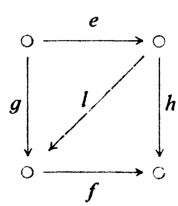
(4) *for any continuous function* g *and any dense embedding* e *the equality* $f = g \,.\, e$ *implies that* e *is a homeomorphism,*

(5) *for any continuous function* g *and any dense compact-extendable map* e *the equality* $f = g \,.\, e$ *implies that* e *is a homeomorphism,*

(6) *for any commutative diagram*



with e *being dense and compact-extendable there exists a unique* l *such that the diagram*



commutes,

(7) *any commutative diagram*

$$\begin{array}{ccc} \circ & \xrightarrow{\;e\;} & \circ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \circ & \xrightarrow[\;m\;]{} & \circ \end{array}$$

for which e is a dense embedding and m is an embedding must be a pullback,

(8) *any commutative diagram*

$$\begin{array}{ccc} \circ & \xrightarrow{\;e\;} & \circ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \circ & \xrightarrow[\;m\;]{} & \circ \end{array}$$

for which e and m are dense and compact-extendable must be a pullback,

(9) *if* $\beta_X : X \to \beta X$ *and* $\beta_Y : Y \to \beta Y$ *denote the Čech-Stone compactification of X and Y, respectively, then the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\;\beta_X\;} & \beta X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \beta f} \\ Y & \xrightarrow[\;\beta_Y\;]{} & \beta Y \end{array}$$

is a pullback.

Proposition 2 (well-known). *The class* $\mathscr{P}$ *of all perfect maps has the following properties*

(1) $\mathscr{P}$ *contains all closed embeddings,*
(2) $\mathscr{P}$ *is closed under composition,*
(3) $\mathscr{P}$ *is closed under inverse images and more generally under pullbacks,*
(4) $\mathscr{P}$ *is closed under multiple pullbacks,*
(5) $\mathscr{P}$ *is closed under products,*
(6) $\mathscr{P}$ *is closed under left cancellation, i.e.,* $f \cdot g \in \mathscr{P}$ *implies* $g \in \mathscr{P}$.

Proposition 3. *Each continuous function has an essentially unique (dense and compact-extendable, perfect) – factorization.*

B. Generalizations. Let $\mathscr{C}$ be a category.

Definition 1. Let $\mathscr{K}$ be a class of objects in $\mathscr{C}$ (resp. a full subcategory of $\mathscr{C}$). A morphism $f : X \to Y$ is called *$\mathscr{K}$-extendable* provided for each $K \in \mathscr{K}$ and each $g : X \to K$ there exists a $\bar{g} : Y \to K$ with $g = \bar{g} . f$.

Denote the class of $\mathscr{K}$-extendable epimorphisms by $E(\mathscr{K})$.

Proposition 4. (1) *$E(\mathscr{K})$ contains all isomorphisms of $\mathscr{C}$.*

(2) *$E(\mathscr{K})$ is closed under composition.*

(3) *$E(\mathscr{K})$ is closed under pushouts.*

(4) *$E(\mathscr{K})$ is left-cancellative with respect to epis, i.e. $f . g \in E(\mathscr{K})$ and g epi implies $g \in E(\mathscr{K})$.*

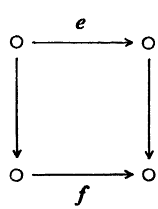
If $\mathscr{C}$ is complete, cocomplete, well-powered, and cowell-powered then:

(5) *For each $\mathscr{C}$-object X there exists a morphism $e_X : X \to \tilde{X}$ in $E(\mathscr{K})$ which is characterized by the fact that for each $e : X \to Y$ in $E(\mathscr{K})$ there exists some f with $e_X = f . e$.*

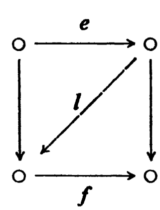
(6) *The full subcategory $\mathscr{K}$ of $\mathscr{C}$ consisting of those $\mathscr{C}$-objects X for which each $e \in E(\mathscr{K})$ is $\{X\}$-extendable is the epireflective hull of $\mathscr{K}$ in $\mathscr{C}$.*

(7) *A class $\mathscr{E}$ of epimorphisms in $\mathscr{C}$ is of the form $E(\mathscr{K})$ for some $\mathscr{K}$ iff $\mathscr{E}$ satisfies the conditions corresponding to (1)–(5).*

Definition 2. a) Let $\mathscr{E}$ be a class of epimorphisms in $\mathscr{C}$. Denote by $M(\mathscr{E})$ the class of all morphisms f such that whenever



commutes and $e \in \mathscr{E}$ then there exists a morphism l such that



commutes.

b) Let $\mathscr{K}$ be a class of objects in $\mathscr{C}$. Then $P(\mathscr{K}) = M(E(\mathscr{K}))$ is called the class of *$\mathscr{K}$-perfect morphisms*.

Proposition 5. (1) *$M(\mathscr{E})$ contains all strong monomorphisms.*

(2) *$M(\mathscr{E})$ is closed under composition.*

(3) *$M(\mathscr{E})$ is closed under pullbacks.*

(4) *$M(\mathscr{E})$ is closed under multiple pullbacks.*

(5) *$M(\mathscr{E})$ is closed under products.*

(6) *$M(\mathscr{E})$ is closed under left-cancellation.*

Theorem. *Let $\mathscr{C}$ be complete, cocomplete, well-powered, and cowell-powered. Let $\mathscr{K}$ be a class of $\mathscr{C}$-objects, $\tilde{\mathscr{K}}$ be its epireflective hull in $\mathscr{C}$, $\mathscr{E} = E(\mathscr{K})$, and $\mathscr{P} = M(\mathscr{E}) = P(\mathscr{K})$. Then:*

(I) *For each $\mathscr{C}$-object A the following conditions are equivalent:*

(1) $A \in \tilde{\mathscr{K}}$,

(2) *if T is terminal then the unique $\mathscr{C}$-morphism $A \to T$ belongs to $\mathscr{P}$,*

(3) *each morphism with domain A belongs to $\mathscr{P}$,*

(4) *for each $\mathscr{C}$-object B the projection $A \times B \to B$ belongs to $\mathscr{P}$.*

(II) *For each $\mathscr{C}$-morphism $f : A \to B$ the following conditions are equivalent:*

(1) $f \in \mathscr{P}$,

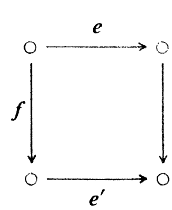
(2) *for each $\mathscr{C}$-object C the morphism $f \times 1_C$ belongs to $\mathscr{P}$,*

(3) *each pullback of f belongs to $\mathscr{P}$,*

(4) *for any $\mathscr{C}$-morphism g and any $e \in \mathscr{E}$ the equality $f = g \,.\, e$ implies that e is an isomorphism.*

In case $\mathscr{C}$ is the category of completely regular spaces and continuous maps and $\mathscr{K}$ contains all compact spaces these conditions are equivalent to

(5) *any commutative diagram*



with $\{e, e'\} \subset E$ is a pullback,

(6) *if* $e_A : A \to \gamma A$ *and* $e_B : B \to \gamma B$ *denote the* $\widetilde{\mathscr{K}}$*-reflections of* A *resp.* B *then the diagram*

$$\begin{array}{ccc} A & \xrightarrow{e_A} & \gamma A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \gamma f} \\ B & \xrightarrow[e_B]{} & \gamma B \end{array}$$

is a pullback.

(III) *For each epimorphism* f *of* $\mathscr{C}$ *the following conditions are equivalent:*

(1) $f \in \mathscr{E}$,

(2) $M(\mathscr{E}) \subset M(\{f\})$.

(IV) *Each* $\mathscr{C}$*-morphism has an essentially unique* $(\mathscr{E}, \mathscr{P})$*-factorization.*

Problem. Characterize those classes $\mathscr{P}$ of $\mathscr{C}$-morphisms for which there exists a class $\mathscr{K}$ of $\mathscr{C}$-objects with $\mathscr{P} = P(\mathscr{K})$.