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A CONTRIBUTION TO THE THEORY OF MODULAR SPACES

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1. In this paper we introduce and investigate some modular spaces and connections between these spaces. In the first part the definition of a modular and a pseudomodular and of a modular space are given. Next, some examples of modular spaces depending on a parameter are given. In the second part of this paper a property of these spaces and connections between them are considered.

1.1. Let a real linear space X be given and let ϱ be a functional defined on X with values $-\infty < \varrho(x) \leqq +\infty$. This functional will be called a *pseudomodular*, if it satisfies the following conditions:

$$\varrho(0) = 0,$$
$$\varrho(-x) = \varrho(x),$$
$$\varrho(\alpha x + \beta y) \leqq \varrho(x) + \varrho(y) \text{ for every } \alpha, \beta \geqq 0,\ \alpha + \beta = 1.$$

If ϱ satisfies the condition

$$\varrho(x) = 0 \text{ if and only if } x = 0$$

instead of condition one, then ϱ is called a *modular*. It is easily seen that if ϱ is a pseudomodular on X, then we have always $\varrho(x) \geqq 0$. Now, we define the *modular space*

$$X_\varrho = \{x : \varrho(\lambda x) \to 0 \text{ as } \lambda \to 0,\ x \in X\}.$$

It is quite obvious that defining modulars in different manners, we obtain various modular spaces (see [2]).

1.2. Let X be a real linear space, and let Ξ be an abstract set. Let $\mathfrak{X}$ be a σ-algebra of subsets of the set Ξ, and let m be a nonnegative measure on $\mathfrak{X}$. We consider an extended real-valued function ϱ defined on $\Xi \times X$, satisfying the following conditions:

1. $\varrho(\xi, x)$ is a pseudomodular in X for almost every $\xi \in \Xi$,
2. if $\varrho(\xi, x) = 0$ for almost every $\xi \in \Xi$, then $x = 0$,
3. $\varrho(\xi, x)$ is measurable in Ξ for every $x \in X$.

By means of this function ϱ we define the following functionals in X:

$$\varrho(x) = \int_{\Xi} p(\xi) \frac{\varrho(\xi, x)}{1 + \varrho(\xi, x)} \, dm ,$$

where $p(\xi)$ is measurable, $0 < p(\xi) < \infty$, $\int_{\Xi} p(\xi) \, dm = 1$,

$$\varrho_0(x) = \operatorname*{supess}_{\xi} \varrho(\xi, x) , \quad \varrho_u(x) = \sup_{\xi} \varrho(\xi, x) .$$

Moreover, let $\mathfrak{M} = \{m_\eta\}$, $\eta \in \mathfrak{Y}$, be a family of nonnegative measures on $\mathfrak{X}$, where $\mathfrak{Y}$ is a set of indices. Then we define

$$\varrho_{\sigma(\mathfrak{M})}(x) = \sup_{\eta} \int_{\Xi} \varrho(\xi, x) \, dm_\eta .$$

In particular, if m_η are absolutely continuous with respect to m, then

$$\varrho_{\sigma(\mathfrak{M})}(x) = \sup_{\eta} \int_{\Xi} a(\xi, \eta) \, \varrho(\xi, x) \, dm ,$$

where the kernel $a(\xi, \eta) \geqq 0$ is measurable in Ξ for every $\eta \in \mathfrak{Y}$. Two special cases will be of importance. In the first one with $a(\xi, \eta) \equiv 1$ we shall write ϱ_s in place of $\varrho_{\sigma(\mathfrak{M})}$, i.e.,

$$\varrho_s(x) = \int_{\Xi} \varrho(\xi, x) \, dm .$$

The second one is obtained taking $\Xi = \langle 0, \infty)$, $\mathfrak{Y} = \langle \eta^*, \infty)$, where $\eta^* > 0$, m is the Lebesgue measure in Ξ, and

$$a(\xi, \eta) = \begin{cases} 1/\eta & \text{for} \quad \xi \leqq \eta , \\ 0 & \text{for} \quad \xi > \eta . \end{cases}$$

Then we shall write ϱ_σ in place of $\varrho_{\sigma(\mathfrak{M})}$, i.e.,

$$\varrho_\sigma(x) = \sup_{\eta \geqq \eta^*} \frac{1}{\eta} \int_{\eta^*}^{\eta} \varrho(\xi, x) \, d\xi .$$

It is easily verified that ϱ, ϱ_0, ϱ_s and ϱ_σ are modulars in X. If $\varrho(\xi, x)$ is a pseudomodular in X for every $\xi \in \Xi$, then ϱ_u is also a modular, and $\varrho_{\sigma(\mathfrak{M})}$ is in general a pseudomodular in X. The respective modular spaces will be denoted by X_ϱ, X_{ϱ_0}, X_{ϱ_s}, X_{ϱ_σ}, X_{ϱ_u}, and $X_{\varrho_{\sigma(\mathfrak{M})}}$. Let us remark, that taking Ξ to be the set of positive integers, $\mathfrak{X}$ the σ-algebra of all subsets of the set Ξ, and mB the number of elements of the set $B \subset \Xi$, $p(\xi) = (\frac{1}{2})^\xi$, then X_ϱ and $X_{\varrho_0} = X_{\varrho_u}$ are countably modulared space and uniformly countably modulared space, respectively (see [1]). Taking also $\mathfrak{Y}$ to be the set of positive integers and defining m_n, $n \in \mathfrak{Y}$, by means of a matrix $A = (a_{ni})$, $a_{ni} \geqq 0$, i.e., $m_n(i) = a_{ni}$, we obtain the space $X_{\varrho_{\sigma(A)}}$ defined in [3].

2. In this section we shall investigate some properties and connections between the above introduced spaces without any further assumptions on X. It is easily observed that

2.1. *We have* $X_{\varrho_u} \subset X_{\varrho 0} \subset X_{\varrho}$.

2.2. *If* $m\Xi < \infty$, *then* $X_{\varrho_0} \subset X_{\varrho_s}$.

This follows from the inequality $\varrho_s(x) \leqq m\Xi \cdot \varrho_0(x)$.

2.3. *If* Ξ *consists of a countable number of pairwise disjoint atoms* $A_1, A_2, \ldots$ *with respect to the measure* m, *and* $\inf_k mA_k > 0$, *then* $X_{\varrho_s} \subset X_{\varrho_u}$.

This is obtained from the inequality $\varrho_s(x) \geqq \inf_k mA_k \cdot \varrho_0(x)$.

2.4. $X_{\varrho_s} \subset X_{\varrho}$.

To prove this inclusion, let us write $A_n = \{\xi : p(\xi) > n\}$. Then $mA_n \to 0$ as $n \to \infty$. Let us choose an $\varepsilon > 0$. Then there exists an integer n such that $\int_{A_n} p(\xi)\, dm < \frac{1}{2}\varepsilon$, and so

$$\varrho(x) < \tfrac{1}{2}\varepsilon + \int_{\Xi \setminus A_n} p(\xi) \cdot \varrho(\xi, x)\, dm \leqq \tfrac{1}{2}\varepsilon + n \cdot \varrho_s(x).$$

Let $x \in X_{\varrho_s}$, then $\varrho_s(\lambda x) \to 0$ as $\lambda \to 0$, and so there exists a $\lambda_\varepsilon > 0$ such that $\varrho_s(\lambda x) < \varepsilon/(2n)$ for $0 < \lambda < \lambda_\varepsilon$. Hence $\varrho(\lambda x) < \varepsilon$ for $0 < \lambda < \lambda_\varepsilon$, and thus $x \in X_\varrho$.

2.5. *An element* $x \in X$ *belongs to* X_ϱ, *if and only if*, $\varrho(\xi, \lambda x) \to 0$ *as* $\lambda \to 0$ *almost everywhere in* Ξ.

Proof. Let $\lambda_k \downarrow 0$ and let us denote

$$h_k(\xi) = p(\xi) \frac{\varrho(\xi, \lambda_k x)}{1 + \varrho(\xi, \lambda_k x)}.$$

Now, let $\varrho(\xi, \lambda x) \to 0$ as $\lambda \to 0$ almost everywhere in Ξ. Then $h_k(\xi) \leqq p(\xi) \cdot \varrho(\xi, \lambda_k x) \to 0$ as $k \to \infty$ and $h_k(\xi) \leqq p(\xi)$. Hence, by Lebesgue dominated convergence theorem, $\int_\Xi h_k(\xi)\, dm \to 0$, i.e., $\varrho(\lambda_k x) \to 0$. Thus $x \in X_\varrho$.

Conversely, let $x \in X_\varrho$, then $\int_\Xi h_k(\xi)\, dm \to 0$ and so $h_k(\xi) \to 0$ in measure m. By the well-known Riesz theorem, $h_{k_i}(\xi) \to 0$ almost everywhere in Ξ, where $\{k_i\}$ is a subsequence of indices. Hence $\varrho(\xi, \lambda_{k_i} x) \to 0$ as $i \to \infty$ almost everywhere in Ξ. Since $\varrho(\xi, \lambda x)$ is a nondecreasing function of $\lambda > 0$, it follows $\varrho(\xi, \lambda x) \to 0$ as $\lambda \to 0$ almost everywhere in Ξ.

From 2.5 it follows immediately that

2.6. *If* $\varrho(\xi, \lambda x) \to 0$ *as* $\lambda \to 0$ *in measure* m, *then* $x \in X_\varrho$.

The converse statement is true under an additional assumption, namely

2.7. *If the measure m is absolutely continuous with respect to the measure $nA = \int_A p(\xi)\,dm$, $A \in \mathfrak{X}$, and $x \in X_\varrho$, then $\varrho(\xi, \lambda x) \to 0$ as $\lambda \to 0$ in measure m.*

Proof. Since $x \in X_\varrho$, by 2.5 we get $\varrho(\xi, \lambda x) \to 0$ as $\lambda \to 0$ almost everywhere with respect to measure m. But the measure n is absolutely continuous with respect to m, and so $\varrho(\xi, \lambda x) \to 0$ as $\lambda \to 0$ almost everywhere with respect to measure n. Since the measure n is finite, this implies convergence $\varrho(\xi, \lambda x) \to 0$ as $\lambda \to 0$ in measure n. Since m is absolutely continuous with respect to n, this implies $\varrho(\xi, \lambda x) \to 0$ as $\lambda \to 0$ in measure m.

Let us remark that the assumption of absolute continuity of m with respect to n in 2.7 cannot be omitted in general. For example, taking Ξ as the set of positive integers, mB as the number of elements of the set $B \subset \Xi$, and $p(\xi) = (\frac{1}{2})^\xi$, the condition $\varrho(\xi, \lambda x) \to 0$ as $\lambda \to 0$ in measure m is equivalent to the condition $x \in X_{\varrho_0}$, and not to the condition $x \in X_\varrho$.

Now, we proceed to investigation of the space $X_{\varrho_{\sigma(\mathfrak{M})}}$; it is convenient to assume absolute continuity of the measures m_η with respect to m, i.e., the modular $\varrho_{\sigma(\mathfrak{M})}(x)$ is defined by means of a kernel $a(\xi, \eta) \geqq 0$ (see 1).

2.8. *Suppose that a sequence of sets $A_k \in \mathfrak{X}$, $k = 1, 2, \ldots$, a sequence of indices $\{\eta_k\}$ and a sequence of numbers $\{M_k\}$ are given such that $\bigcup_{k=1}^{\infty} A_k = \Xi$ and $a(\xi, \eta_k) > M_k$ for every $\xi \in A_k$. Then $X_{\varrho_{\sigma(\mathfrak{M})}} \subset X_\varrho$.*

Proof. Let $x \in X_{\varrho_{\sigma(\mathfrak{M})}}$, then $\int_\Xi a(\xi, \eta)\,.\,\varrho(\xi, \lambda x)\,dm \to 0$ as $\lambda \to 0$ uniformly in $\mathfrak{Y}$. Taking $\lambda_i \downarrow 0$ and choosing an $\varepsilon > 0$ we have

$$M_k \int_{A_k} \varrho(\xi, \lambda_i x)\,dm \leqq \int_{A_k} a(\xi, \eta_k)\,.\,\varrho(\xi, \lambda_i x)\,dm < \varepsilon$$

for any k and for i sufficiently large. Hence $\varrho(\xi, \lambda_i x) \to 0$ in measure in the set A_k. Since $\varrho(\xi, \lambda x)$ is a nondecreasing function of $\lambda > 0$, the well-known Riesz theorem implies $\varrho(\xi, \lambda_i x) \to 0$ as $i \to \infty$ almost everywhere in A_k. Thus $\varrho(\xi, \lambda x) \to 0$ as $\lambda \to 0$ almost everywhere in Ξ, and so according to 2.5, $x \in X_\varrho$.

2.9. *If $\sup_\eta \int_\Xi a(\xi, \eta)\,dm < \infty$, then $X_{\varrho_0} \subset X_{\varrho_{\sigma(\mathfrak{M})}}$.*

This result follows from the inequality

$$\varrho_{\sigma(\mathfrak{M})}(x) \leqq \sup_\eta \int_\Xi a(\xi, \eta)\,dm\,.\,\varrho_0(x)\,.$$

2.10. *If* $\sup_{\eta} \operatorname{supess}_{\xi} a(\xi, \eta) < \infty$, *then* $X_{\varrho_s} \subset X_{\varrho_{\sigma(\mathfrak{M})}}$.

This follows from the inequality

$$\varrho_{\sigma(\mathfrak{M})}(x) \leqq \sup_{\eta} \operatorname{supess}_{\xi} a(\xi, \eta) \cdot \varrho_s(x) .$$

Remark. Let us note that the results obtained in this paper are generalizations of some results of [3], taking the set of natural numbers as Ξ, the σ-algebra of all subsets of Ξ as $\mathfrak{X}$, the measure mB defined as the number of elements of the set $B \subset \Xi$, and $p(\xi) = (\frac{1}{2})^{\xi}$.

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