Toposym 3

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When categories of presheaves are binding

In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 447--450.

Persistent URL: `http://dml.cz/dmlcz/700733`

WHEN CATEGORIES OF PRESHEAVES ARE BINDING[1])

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Following [8], a category is called *binding* if every category of universal algebras and homomorphisms can be fully embedded into it. If K is binding then every small category – and, under the assumption of non-existence of a proper class of measurable cardinals, every concrete category – admits a full embedding into K ([6], [9]). Particularly, every semigroup with unity can be represented as the semigroup of all endomorphisms of an object of K. Categories with the last property will be called *semibinding*.

While a lot of algebraic categories are binding ([4], [5], [11], [12], [13]), topological categories are often not even semibinding. Every semigroup of all continuous mappings from a topological space into itself contains idempotents (constant mappings) and therefore a non-trivial semigroup without non-identical idempotents cannot be represented in a topological category with continuous mappings as morphisms. Also the category of Hausdorff spaces and local homeomorphisms is not semibinding ([10]).

In the present note we give a certain criterion how far a category is from being binding. This is described by presenting the class P_K of all partially ordered sets k such that the category K^k of presheaves in K over k is binding. If K has an initial or a terminal object (all categories considered here have them both, namely, the empty space and the one-point space) then K is binding if and only if P_K is the class of all non-void partially ordered sets. Thus, roughly speaking , the bigger P_K is, the nearer K is to being binding. Analogously, denote S_K the class of all non-void partially ordered sets k such that K^k is semibinding.

The aim of this note is to describe the classes P_K (or S_K) for some categories familiar in topology. In particular the categories mentioned in Theorem 1 are binding.

The full text with proofs will be sent to Czechoslovak Math. J. The proofs of all theorems except Theorem 4 use the space M_1 from [2].

Definitions and conventions. The symbol $\dot{\subset}$ will be used for full embedding. As usual, every partially ordered set is considered as a thin category ($a \leqq b$ if and only if there exists a morphism from a to b).

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[1]) Preliminary communication.

If K is a category, k a partially ordered set, denote by K^k the category of presheaves in K over k (= covariant functors from the thin category k to K) and their transformations.

Denote by $\mathbf{P}$ the class of all non-void partially ordered sets. Denote P_K (or S_K) the class of all $k \in \mathbf{P}$ such that K^k is binding (or semibinding, respectively).

The following three propositions are evident:

Proposition 1. *Let a category K have an initial or a terminal object. Then $P_K = \mathbf{P}$ (or $S_K = \mathbf{P}$) if and only if K is binding (or semibinding, respectively).*

Proposition 2. *If $K \dot{\subset} H$ then $P_K \subset P_H$ and $S_K \subset S_H$.*

Proposition 3. *Let $K \dot{\subset} L \dot{\subset} H$. If $P_K = P_H$ (or $S_K = S_H$) then $P_K = P_L = P_H$ (or $S_K = S_L = S_H$, respectively).*

Theorem 1. *$P_K = S_K = \mathbf{P}$ for the following types of categories:*

1) *All subcategories of the category of metric spaces and open proximally continuous mappings containing the category of metric spaces and open Lipschitz mappings with bound* 1.

2) *All subcategories of the category of topological spaces and open continuous mappings containing the category of T_1-spaces and open local homeomorphisms.*

3) *All subcategories of the category of T_1-spaces and continuous locally one-to-one mappings containing all local homeomorphisms.*

Convention. If $G \subset \mathbf{P}$, put

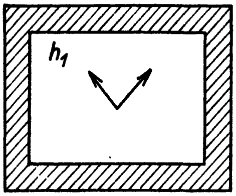
$$\operatorname{gen} G = \{k \in \mathbf{P};\ (\exists h \in G)(h \dot{\subset} k)\}\,.$$

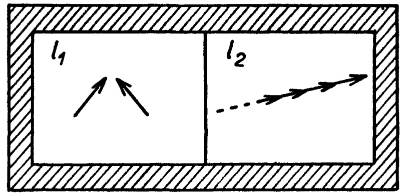
Put $A = \operatorname{gen}\{h_1\}$, $B = \operatorname{gen}\{l_1\}$, $C = \operatorname{gen}\{l_1, l_2\}$, $D = \operatorname{gen}\{k_1, \ldots, k_{35}\}$ (see the figure on page 449).

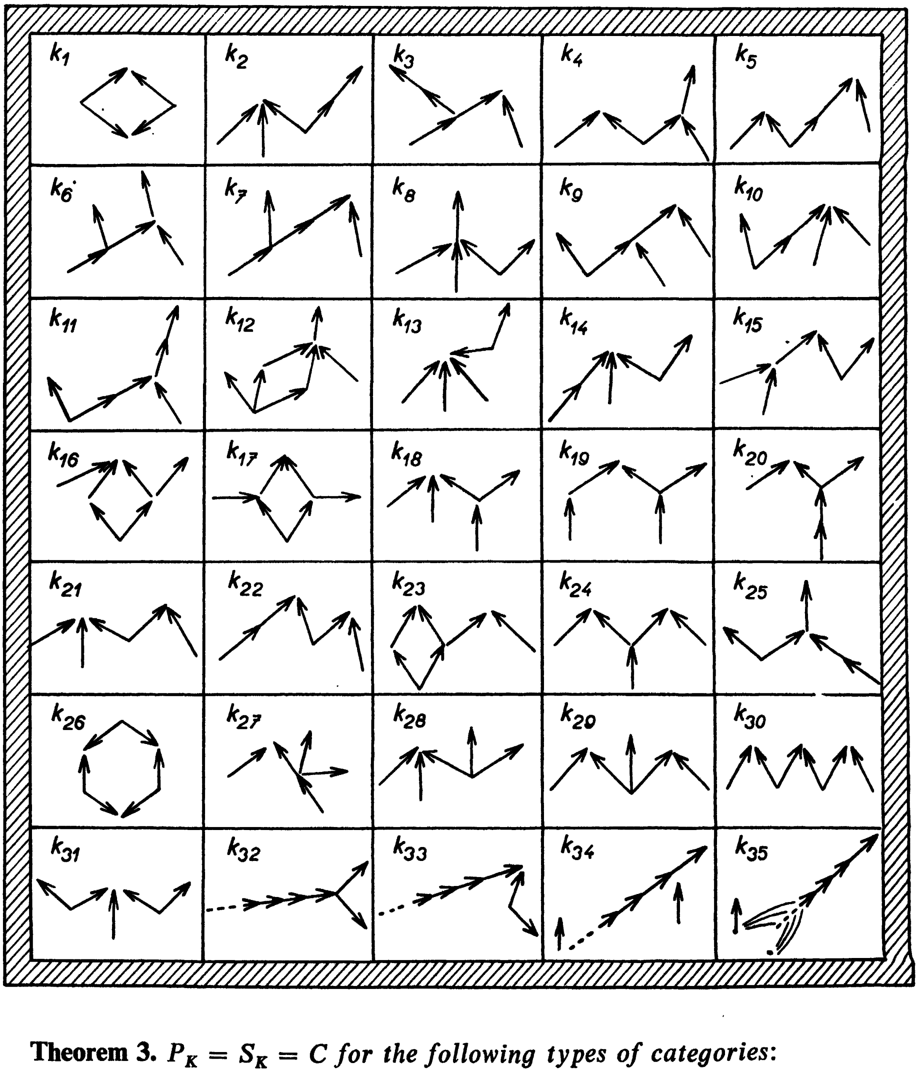
Theorem 2. *$P_K = S_K = A$ for*

1) *all subcategories of the category of Hausdorff spaces and locally one-to-one continuous mappings, containing either the category of compact Hausdorff spaces and local homeomorphisms or the category of metrizable spaces and local homeomorphisms,*

2) *all subcategories of the category of Hausdorff spaces and open continuous mappings, containing either the category of compact Hausdorff spaces and open local homeomorphisms or the category of metrizable spaces and open local homeomorphisms.*







Theorem 3. $P_K = S_K = C$ *for the following types of categories:*

1) *All full subcategories of the category of topological (or proximity or uniform) spaces and continuous (or proximally continuous or uniformly continuous, respectively) mappings, containing all metrizable spaces.*

2) *All subcategories of the category of metric spaces and continuous mappings containing the category of metric spaces and Lipschitz mappings with bound* 1.

Theorem 4. $P_{\mathscr{S}} = D$, *where* $\mathscr{S}$ *is the category of sets and mappings.*

The following two theorems are consistent with the set-theory:

Theorem 5. $P_K = B$, *where* K *is the category of compact Hausdorff spaces and continuous mappings.*

Theorem 6. $P_K = C$ *for all full subcategories* K *of the category of* T_1*-spaces and continuous closed mappings containing all locally compact* σ*-compact Hausdorff spaces.*

Note. Every compact Hausdorff space with the first axiom of countability has the power $\leqq 2^{\aleph_0}$ ([1]). Consequetly $P_K = \emptyset$ for a category of these spaces with any choice of morphisms such that all homeomorphisms are included.

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