

Toposym 3

Djuro R. Kurepa

Factorials and the general continuum hypothesis

In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 281--282.

Persistent URL: <http://dml.cz/dmlcz/700732>

Terms of use:

© Institute of Mathematics AS CR, 1972

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

FACTORIALS AND THE GENERAL CONTINUUM HYPOTHESIS

D. KUREPA

Beograd

1. Number $n!$

1.1. Definition. For any (cardinal or ordinal) number n we define the *right factorial* $n!$ as the cardinality $kS!$ of the set $S!$ of all permutations (\equiv one-to-one mappings of S onto S) of S , S being any set having kn (\equiv card n) members; we put $kn = n$ for every cardinality n .

1.2. Right factorial hypothesis.

RFH $n! = 2^{kn}$ for every transfinite (cardinal or ordinal) number n .

1.3. Theorem. $Z \Rightarrow$ RFH, (Z denoting the choice axiom).

(See [1], [1^a] Th. 2.2; in particular we proved that for transfinite cardinalities $n^2 = n \Rightarrow n! = 2^n$; therefore [since $\aleph_\alpha^2 = \aleph_\alpha$] RFH holds for transfinite ordinal numbers.)

1.4. Problem. Does RFH \Rightarrow Z?

2. Left factorials

2.1. In [4] we defined the left factorial $!n$ by

$$(2.1) \quad !n = \sum m!, \quad 0 \leq m < n$$

for any (cardinal or ordinal, finite or transfinite) number n .

2.2. We also proved ([4] Th. 6.2. ((0000))) that the GCH (general continuum hypothesis) implies the following *left factorial hypothesis*

LFH $!n = n$ for every transfinite cardinality n .

2.3. On the other side, GCH \Rightarrow Z (Sierpiński [5]). In other words

$$(2.2) \quad \text{GCH} \Rightarrow \text{LFH} \wedge Z.$$

2.4. *The converse of (2.2) also holds.*

Proof. In the opposite case, there would be a pair of transfinite cardinalities (n, r) such that

$$(2.3) \quad n < r < 2^n.$$

Now, $n < r$ implies

$$(2.4) \quad n! \leq !r$$

because $n!$ is a summand for $!r$. On the other hand

$$(2.5) \quad Z \Rightarrow n! = 2^n$$

for any transfinite cardinality n (Theorem 1.3), and (2.4) would yield $2^n \leq !r =$ (by LFH) $= r$, i.e., $2^n \leq r$, contradicting (2.3).

Consequently, one has the following

2.5. Theorem. $GCH \Leftrightarrow LFH \wedge Z$.

2.6. Problem. Does $LFH \Rightarrow GCH$?

In connection with 2.5 and 2.6 one has the following

2.7. Theorem. $GCH \Leftrightarrow 2^{\aleph} = \aleph$ *identically for transfinite cardinalities* \aleph (cf. Tarski [6], L. 9a), p. 194), *where, by definition,* $2^{\aleph} = \sum_{m < \aleph} 2^m$ (see Tarski [6] Def 4).

References

- [1] *D. Kurepa:* O faktorijelama konačnih i beskonačnih brojeva. Rad. Jugosl. Akad. Znan. Umjet. 296 (1953), 105–122.
- [1a] *D. Kurepa:* Über die Faktoriellen der endlichen und unendlichen Zahlen. Bull. Internat. Acad. Yougoslave Sci. 12 (1954), 51–64.
- [2] *D. Kurepa:* Über das Auswahlaxiom. Math. Ann. 126 (1953), 381–384.
- [3] *D. Kurepa:* Sull'ipotesi del continuo. Rendiconti del Seminario Matematico, Torino 18 (1958–1959), 11–20.
- [4] *D. Kurepa:* Factorials of cardinal numbers and trees. Glasnik mat.-fiz. astr. 19 (1964), 7–21.
- [5] *W. Sierpiński:* L'hypothèse généralisée du continu et l'axiome de choix. Fund. Math. 34 (1947), 1–5.
- [6] *A. Tarski:* Sur les classes d'ensembles closes par rapport à certaines opérations élémentaires. Fund. Math. 16 (1930), 181–304.