FACTORIALS AND THE GENERAL CONTINUUM HYPOTHESIS

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1. Number n!

1.1. Definition. For any (cardinal or ordinal) number n we define the *right factorial* $n!$ as the cardinality k$S!$ of the set $S!$ of all permutations ($\equiv$ one-to-one mappings of S onto S) of S, S being any set having kn ($\equiv$ card n) members; we put k$n = n$ for every cardinality n.

1.2. Right factorial hypothesis.

RFH $n! = 2^{kn}$ for every transfinite (cardinal or ordinal) number n.

1.3. Theorem. Z $\Rightarrow$ RFH, (Z *denoting the choice axiom*).

(See [1], [1ᵃ] Th. 2.2; in particular we proved that for transfinite cardinalities $n^2 = n \Rightarrow n! = 2^n$; therefore [since $\aleph_\alpha^2 = \aleph_\alpha$] RFH holds for transfinite ordinal numbers.)

1.4. Problem. Does RFH $\Rightarrow$ Z?

2. Left factorials

2.1. In [4] we defined the left factorial $!n$ by

$$!n = \sum m!\,, \quad 0 \leqq m < n \tag{2.1}$$

for any (cardinal or ordinal, finite or transfinite) number n.

2.2. We also proved ([4] Th. 6.2. ((0000)) that the GCH (general continuum hypothesis) implies the following *left factorial hypothesis*
LFH $!n = n$ for every transfinite cardinality n.

2.3. On the other side, GCH $\Rightarrow$ Z (Sierpiński [5]). In other words

$$\text{GCH} \Rightarrow \text{LFH} \wedge \text{Z}\,. \tag{2.2}$$

2.4. *The converse of* (2.2) *also holds.*

Proof. In the opposite case, there would be a pair of transfinite cardinalities (n, r) such that

$$(2.3) \qquad n < r < 2^n .$$

Now, $n < r$ implies

$$(2.4) \qquad n! \leqq !r$$

because $n!$ is a summand for $!r$. On the other hand

$$(2.5) \qquad Z \Rightarrow n! = 2^n$$

for any transfinite cardinality n (Theorem 1.3), and (2.4) would yield $2^n \leqq !r =$ $=$ (by LFH) $= r$, i.e., $2^n \leqq r$, contradicting (2.3).

Consequently, one has the following

2.5. Theorem. GCH $\Leftrightarrow$ LFH $\wedge$ Z.

2.6. Problem. Does LFH $\Rightarrow$ GCH?

In connection with 2.5 and 2.6 one has the following

2.7. Theorem. GCH $\Leftrightarrow 2^{\underline{n}} = n$ *identically for transfinite cardinalities* n (cf. Tarski [6], L. 9a), p. 194), *where, by definition,* $2^{\underline{n}} = \sum_{m<n} 2^m$ (see Tarski [6] Def 4).

References

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