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SOME REMARKS CONCERNING THE THEORY OF SHAPE IN ARBITRARY METRIZABLE SPACES

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The classical notion of the homotopy type (introduced by W. Hurewicz, [4], p. 125) allows to classify the spaces from the point of view of their most important global topological properties, called homotopy properties. Let us recall the definition of this notion:

Two spaces X, Y have the same *homotopy type* (notation: $X \underset{h}{\simeq} Y$) if there exist two maps (i.e. continuous functions) $f: X \to Y$ and $g: Y \to X$ such that $gf: X \to X$ and $fg: Y \to Y$ are homotopic to the identities $i_X: X \to X$ and $i_Y: Y \to Y$ respectively.

If only the relation $fg \simeq i_Y$ holds, then one says (following J. H. C. Whitehead, [6], p. 1133) that X *homotopically dominates* Y and we write $X \underset{h}{\geq} Y$.

Both notions of the homotopy type and of the homotopy domination are important tools in the study of global properties of spaces with a rather regular local topological structure as polyhedra or — more generally — as absolute neighborhood retracts for metric spaces, i.e. ANR$(\mathfrak{M})$-spaces. However, for spaces with a more complicated local structure, the family of maps of one space into another can be too limited to give a basis for a reasonable classification of spaces from the point of view of their global properties.

The endeavor to avoid this difficulty is the origin of the theory of shape. Instead of the notion of maps, one uses in it a more elastic notion of the fundamental sequences, which in the most important case of compacta (that is of metric compact spaces) may be defined as follows:

Let X be a compactum lying in a space $M \in \mathrm{AR}(\mathfrak{M})$ and Y be a compactum lying in another space $N \in \mathrm{AR}(\mathfrak{M})$. By a fundamental sequence $\boldsymbol{f}: X \to Y$ one understands a triple $\{f_k, X, Y\}_{M,N}$ consisting of a sequence of maps $f_k: M \to N$ and of the compacta X, Y satisfying the following condition:

(I) *For every neighborhood V of Y (in N) there is a neighborhood U of X (in M) such that $f_k/U \simeq f_{k+1}/U$ in V for almost all k.*

In particular, if $X = Y$, $M = N$ and if f_k denotes the identity map $i: M \to M$ for every $k = 1, 2, \ldots$, then $\{f_k, X, X\}_{M,M}$ is a fundamental sequence, which we denote by $\boldsymbol{i}_{X,M}$.

If $\boldsymbol{f} = \{f_k, X, Y\}_{M,N}$ and $\boldsymbol{g} = \{g_k, Y, Z\}_{N,P}$ are two fundamental sequences, then $\boldsymbol{gf} = \{g_k f_k, X, Z\}_{M,P}$ is also a fundamental sequence. Two fundamental sequences

$f = \{f_k, X, Y\}_{M,N}$ and $f' = \{f'_k, X, Y\}_{M,N}$ are said to be *homotopic* (notation: $f \simeq f'$) if

(II) *For every neighborhood V of Y (in N) there is a neighborhood U of X (in M) such that $f_k/U \simeq f'_k/U$ in V for almost all k.*

By the *shape* $\mathrm{Sh}(X)$ of a compactum X one understands the collection of all compacta Y such that there exist two fundamental sequences $f: X \to Y$, $g: Y \to X$ such that $gf \simeq i_{X,M}$ and $fg \simeq i_{Y,N}$. We write $\mathrm{Sh}(X) \geqq \mathrm{Sh}(Y)$ if there exist two fundamental sequences $f: X \to Y$ and $g: Y \to X$ such that $fg \simeq i_{Y,N}$. One sees easily that the choice of spaces $M, N \in \mathrm{AR}(\mathfrak{M})$, and also the choice of the embeddings of X and of Y into M and N respectively is immaterial. The properties of a compactum X depending only on the shape $\mathrm{Sh}(X)$ are called *shape invariants*. One shows that the most important global topological properties of compacta, as the homology groups (in the sense of Vietoris or of Čech, but not the singular homology groups) are shape invariants. The homotopy groups (in the classical sense) do not belong to shape invariants, but by a slight modification of their definitions, one obtains the so called *fundamental groups* (see [1], p. 251) which are shape invariants. Also cohomology groups ([5], p. 54) and cohomotopy groups ([3], p. 81) belong to shape invariants.

Let us add that for compact ANR-spaces, the notion of the shape is the same as the notion of the homotopy type, i.e. if $X, Y \in \mathrm{ANR}$ then $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ if and only if $X \underset{h}{\simeq} Y$, and $\mathrm{Sh}(X) \geqq \mathrm{Sh}(Y)$ if and only if X homotopically dominates Y.

The theory of shape of compacta has been already developed, due to work of T. A. Chapman, S. Godlewski, D. Henderson, W. Holsztyński, S. Mardešić, M. Moszyńska, H. Patkowska, J. Segal and others. However, the attempts to extend it onto arbitrary metrizable spaces are only at the beginning. The purpose of this note is to point out some difficulties which appear in this way.

By a theorem of K. Kuratowski and M. Wojdysławski ([7], p. 186), every metric space X can be embedded as a closed subset of an $\mathrm{AR}(\mathfrak{M})$-space M. This fact allows to replace in the definition of the fundamental sequences and in the definition of their homotopy given by conditions (I) and (II) the hypothesis that the considered spaces X and Y are compacta by a weaker one, viz. that they are arbitrary metric spaces. However, it seems, that the definitions obtained in this way are not satisfactory, because they do not constitute a sufficient base for the proof that the homology groups (in the sense of Vietoris, with compact carriers) belong to shape invariants. But this difficulty is not serious, because for compacta, conditions (I) and (II) are equivalent to the following conditions:

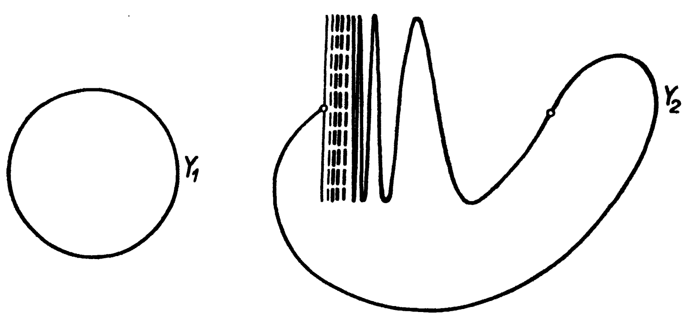
(I′) *For every compactum $A \subset X$ there is a compactum $B \subset Y$ such that for every neighborhood V of B in N there exists a neighborhood U of A in M such that $f_k/U \simeq f_{k+1}/U$ in V for almost all k.*

(II′) *For every compactum $A \subset X$ there is a compactum $B \subset Y$ such that for every neighborhood V of B in N there exists a neighborhood U of A in M such that $f_k/U \simeq f'_k/U$ in V for almost all k.*

One can show by examples that conditions (I) and (I′), though equivalent for compacta, are independent from each other in the case of arbitrary metric spaces. In this general case, the condition (I′) seems to be more important than the condition (I) because, by the definition of the shape based on it, the homology groups, and also the fundamental groups are shape invariants.

In one of my notes [2] I develop the theory of shape for arbitrary metrizable spaces, starting from the definition of the fundamental sequences as triples $\{f_k, X, Y\}_{M,N}$ satisfying both conditions (I), (I′) and using the definition of the homotopy based on both conditions (II) and (II′). Let us call the theory of shape obtained in this way the *strong shape theory*, or shortly, *S-shape theory*, and let us call the fundamental sequences in this sense — the *S-sequences*, their homotopy — the *S-homotopy* and the corresponding notion of shape of a space X, the *S-shape* $\mathrm{Sh}_S(X)$. One shows that in this S-shape theory (which for compacta is the same as the usual shape theory) many results extend from compacta onto arbitrary metrizable spaces. In particular, one shows that if $X, Y \in \mathrm{ANR}(\mathfrak{M})$ then the relation $\mathrm{Sh}_S(X) = \mathrm{Sh}_S(Y)$ is equivalent to the homotopy equivalence of X and Y, and the relation $\mathrm{Sh}_S(X) \geqq \mathrm{Sh}_S(Y)$ is equivalent to the homotopy domination of Y by X.

Unfortunately the S-shape theory is not always adequate to our intuition. The following simple example, due to S. Nowak, shows that the S-shape of the Cartesian product $X \times Y$ of two spaces X and Y is not determined by the S-shapes of X and of Y. In fact, let X denote the space of all natural numbers (with the usual, discrete topology) and let Y_1 be a circle and Y_2 be the so called "Polish circle" (that is the union of the closure C of the diagram of the function $y = \sin \pi/x$ for $0 < x < 1$ and of an arc with endpoints (0, 0) and (1, 0) and the interior disjoint with C). It is clear that $\mathrm{Sh}_S(Y_1) = \mathrm{Sh}_S(Y_2)$, but it is easy to show that $\mathrm{Sh}_S(X \times Y_1) \neq \mathrm{Sh}_S(X \times Y_2)$.



One obtains another extension of the theory of shape onto all metrizable spaces if one bases the definition of the fundamental sequences only on the condition (I′) and the definition of their homotopy on the condition (II′). Thus we get a theory which we call the *weak shape theory*, or shortly the *W-shape theory*, dealing with the corresponding notion of the *W-shape*. For compacta the W-theory is the same as the usual shape theory and one obtains in it generalizations of many theorems concerning the shapes, from compacta onto arbitrary metrizable spaces. In particular, one proves that the W-shape of the Cartesian product $X \times Y$ depends only on the W-shapes of X and of Y. As I have already mentioned, an analogous statement is not true in the S-theory.

Thus the W-theory has some advantages in comparison with the S-theory. However, it is rather doubtful if the W-theory satisfies all our expectations. In particular, we do not know if it is a direct extension of the classical homotopy theory (limited to ANR $(\mathfrak{M})$-spaces), because the problem whether two ANR $(\mathfrak{M})$-spaces with the same W-shape are necessarily homotopically equivalent, remains still open. As I have already mentioned, an analogous proposition holds in the S-theory.

Let us confront some features of both theories: the S-theory and the W-theory, with some postulates which one may expect from a reasonable extension of the theory of shape onto all metrizable spaces:

Some postulates for an extension of the theory of shape onto all metrizable spaces	S-theory	W-theory
Coincidence with the usual shape-theory for compacta.	+	+
If $X, Y \in$ ANR $(\mathfrak{M})$ then X and Y have the same shape iff $X \underset{h}{\simeq} Y$.	+	?
Homology groups (of Vietoris with compact carriers) are shape-invariants.	+	+
Shape of $X \times Y$ depends only on the shape of X and of Y.	—	+
Shape of the suspension of X depends only on the shape of X.	+	+
Fundamental groups (of pointed spaces) are shape invariants.	+	+
If $(Z, z_0) = (X, x_0) \underset{\text{top}}{+} (Y, y_0)$ then the shape of (Z, z_0) depends only on the shapes of (X, x_0) and of (Y, y_0).	+	+

It follows by this schedule that S-theory does not satisfy one of our postulates, and that probably a similar situation is with the W-theory. Thus, for the present we do not have a unique extension of the theory of shape onto all metrizable spaces consistent with the geometric intuition. One can suspect that the situation in the theory of shape is similar to the well-known situation in the theory of dimension, where for the class of metric separable spaces one has a unique reasonable theory of dimension, but if we pass to more general spaces then it is necessary to develop several different theories of dimension. So in the theory of shape, we have a unique theory for compacta, but for arbitrary metric spaces it seems to be necessary to develop several different theories, in particular the weak and the strong theory of shape.

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