Toposym 3

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A CATEGORICAL GENERALIZATION OF COMPLETELY HAUSDORFF SPACES

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A topological space X is well known to be called a completely Hausdorff space if and only if for any two distinct points $x, y \in X$ there exists a continuous function f from X to the space $\mathbf{R}$ of real numbers with $f(x) \neq f(y)$. If we substitute the space $\mathbf{R}$ of real numbers in the definition of a completely Hausdorff space by the discrete space D_2 consisting of exactly two points we obtain the class of all spaces whose quasi-components consist only of a single point. Taking the Sierpinski space S (that means the topological space consisting of exactly two points and three open sets) instead of $\mathbf{R}$ we obtain the class of all T_0-spaces. Now let us consider a whole class $\mathscr{E}$ of (non-void) topological spaces and let us define a class $Q\mathscr{E}$ of topological spaces in the following way:

$X \in Q\mathscr{E} \Leftrightarrow$ For any two distinct points $x, y \in X$ there exists a space $E \in \mathscr{E}$ and a continuous map $f : X \to E$ with $f(x) \neq f(y)$.

Let us choose for $\mathscr{E}$ the class of all spaces with the topology of finite complements. Then $Q\mathscr{E}$ consists of exactly all T_1-spaces. It is impossible to find a space E such that $Q\{E\} = \{T_1\text{-spaces}\}$. This is an immediate consequence of a result of Herrlich that for each T_1-space X there is a regular T_1-space Y consisting of at least two points such that each continuous function f from Y into X is constant.

Now let $\mathscr{C}$ be a category and $\mathscr{A}$ a (full) subcategory. Then we define a full subcategory $Q_{\mathscr{C}}\mathscr{A}$ of $\mathscr{C}$ by defining the object class $|Q_{\mathscr{C}}\mathscr{A}|$ of $Q_{\mathscr{C}}\mathscr{A}$ by

$|Q_{\mathscr{C}}\mathscr{A}| = \{X \mid$ For any two distinct morphisms $\alpha, \beta : Z \to X$ there exists an object $A \in |\mathscr{A}|$ and a morphism $f : X \to A$ with $f \circ \alpha \neq f \circ \beta\}$.

(If we take for $\mathscr{C}$ the category of topological spaces (and continuous maps) and define $\mathscr{A}$ by $|\mathscr{A}| = \mathscr{E}$, then we obtain

$$|Q_{\mathscr{C}}\mathscr{A}| = Q\mathscr{E}$$

in the sense defined above.)

It turns out that $Q_{\mathscr{C}}\mathscr{A}$ is isomorphically closed as well as closed under formation of products (in the categorical sense) and extremal subobjects (that means subspaces in the topological case). If $\mathscr{C}$ is a "nice" category we may conclude that $Q_{\mathscr{C}}\mathscr{A}$ is an epireflective subcategory of $\mathscr{C}$, but it does not generally coincide with the epireflective hull $R_{\mathscr{C}}\mathscr{A}$ of $\mathscr{A}$ in $\mathscr{C}$ (i.e., the smallest epireflective [full and isomorphically closed] subcategory of $\mathscr{C}$ containing $\mathscr{A}$). But *if in addition $\mathscr{C}$ is balanced (that means every $\mathscr{C}$-bimorphism is a $\mathscr{C}$-isomorphism), we get the result*

$$(*) \qquad Q_{\mathscr{C}}\mathscr{A} = R_{\mathscr{C}}\mathscr{A} .$$

Take for $\mathscr{C}$ the category of compact T_2-spaces (and continuous maps) and define $\mathscr{A}$ by $|\mathscr{A}| = \{D_2\}$; then *we obtain from (*) the well known fact that a compact T_2-space is zero-dimensional if and only if each of its quasi-components consists of a single point.*

For a "nice" category (that means complete, locally and colocally small) the following theorem is valid:

Theorem. *The following statements are equivalent:*

(1) $X \in |Q_{\mathscr{C}}\mathscr{A}|$.

(2) *The reflection map* $r_x : X \to X_{R_{\mathscr{C}}\mathscr{A}}$ *is a monomorphism.*

(3) *X is subobject of a product of $\mathscr{A}$-objects (in $\mathscr{C}$).*

Corollary. *$Q_{\mathscr{C}}\mathscr{A}$ is the smallest (full and isomorphically closed) subcategory of $\mathscr{C}$ containing $\mathscr{A}$ and being closed under formation of subobjects and products.*

Example. The category of completely Hausdorff spaces (and continuous maps) is the smallest subcategory of the category $\mathscr{T}$ of topological spaces (and continuous maps) containing the space $\mathbf{R}$ of real numbers and being closed under formation of subobjects and products in $\mathscr{T}$.

Problem. If $\mathscr{C}$ is not balanced, find a "nice" condition on X such that $X \in$ $\in |Q_{\mathscr{C}}\mathscr{A}|$ together with this condition implies $X \in |R_{\mathscr{C}}\mathscr{A}|$.

References

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