ON A METHOD WHICH LEADS TO EXTREMALLY DISCONNECTED COVERS

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Recently Hager [4] has found a simple method (using Kuratowski-Zorn Lemma) to get for each compact Hausdorff space the associated extremally disconnected cover (the construction due originally to Gleason [3] and simplified by Rainwater [8]). The aim of this note is to show that this method may be applied in order to get extremally disconnected covers for arbitrary Hausdorff spaces.

Throughout this text all the spaces are assumed (this is in fact not always necessary) to be Hausdorff and all the maps are assumed to be continuous.

A map $p : Y \xrightarrow{\text{onto}} X$ will be said to be *irreducible* iff for each closed subset F of Y we have $\mathrm{Cl}\, p(F) \neq X$ whenever $F \neq Y$; in the case of compact Hausdorff spaces this notion coincides with the usual one. Irreducible maps $Y \to X$ will be said also to be *covers* of X, and in the case when Y is extremally disconnected, *extremally disconnected covers* of X.

It was shown by L. Rudolf and the author in [7] that

1. *If $p : Y \to X$ is irreducible, then for each V regularly open in Y there exists a U regularly open in X, such that* $V = \mathrm{Int}\, p^{-1}(\mathrm{Cl}\, U) = \mathrm{Int}\,\mathrm{Cl}\, p^{-1}(U)$, *and this determines U for a given V.*

The above lemma established a one-to-one correspondence, $p_* : U \to \mathrm{Int}\, p^{-1}(\mathrm{Cl}\, U)$, between the families of regularly open subsets of X and Y, which preserves the Boolean operations. It is easy to check that if u is an ultrafilter in X (we consider only filters consisting of regularly open subsets), then $p_* u = \{p_*(U) : U \in u\}$ is an ultrafilter in Y. Thus the underlying set of Y may be regarded as a subset of the family $R(X)$ of all ultrafilters in X, and consequently, every class of non-homeomorphic covers of X is a set. Let $\mathrm{Cov}\, X$ be a fixed set of covers of X such that each cover of X is homeomorphic to one of $\mathrm{Cov}\, X$.

The following cancellation law may be found in [7] (see [8] for the compact case):

2. *If $p : Y \to X$ is irreducible and $h : Y \to Y$ is such that $p \circ h = p$, then h is the identity.*

This leads to a corollary that we get a partial ordering in Cov X if we set $p \prec q$ for covers $p : Y \to X$ and $q : Z \to X$ whenever there exists $h : Z \to Y$ such that $p \circ h = q$.

Let R, $R \subset R(X)$, be a set of ultrafilters in X such that for each x in X there exists a u in R such that $x \in \bigcap\{\mathrm{Cl}\, U : U \in u\}$; in particular, the whole $R(X)$ satisfies this condition. A cover $p : Y \to X$ will be said to be an *R-cover* if for each u in R there exists a y in Y such that $y \in \bigcap\{\mathrm{Cl}\, p_*(U) : U \in u\}$; then $p(y)$ is the only point in $\bigcap\{\mathrm{Cl}\, U : U \in u\}$. Let $\mathrm{Cov}_R X$ be the set of all R-covers of X.

Let us note an easy fact (cf. [3]):

3. *If V is regularly open in Y, then the projection $(Y - V) + \mathrm{Cl}\, V \to Y$ is an R-cover for any R, $R \subset R(Y)$*; here "+" stands for the free union. The projection is not a homeomorphism unless V is closed-open. This leads to the conclusion that the maximal elements in $\mathrm{Cov}_R X$ are extremally disconnected covers. To examine the set $\mathrm{Cov}_R X$ with respect to maximal elements, note first that

4. *If $p : Y \to X$ and $q : Z \to X$ are R-covers, $h : Z \to Y$ is such that $p \circ h = q$ and $u \in R$, then $h(z) = y$, whenever $\{y\} = \bigcap\{\mathrm{Cl}\, p_*(U) : U \in u\}$ and $\{z\} = \bigcap\{\mathrm{Cl}\, q_*(U) : U \in u\}$.*

The proof consists in the calculation which makes use of Assertion 1, that $h^{-1}(y) = \bigcap\{\mathrm{Cl}\, q_*(U) : U \in u\}$ (cf. [7], p. 29, for a similar calculation in a more general situation).

Now we shall prove

5. *The set $\mathrm{Cov}_R X$ is closed with respect to inverse limits of directed systems.*

Proof. Let P be a directed system of R-covers of X. Let Z be the inverse limit space of the system and let $p_X : Z \to X$ be the projection.

The projection p_X is onto. To see this, let $x \in X$. Let u in R be such that $\{x\} = \bigcap\{\mathrm{Cl}\, U : U \in u\}$. If Y is a space of the system P, then let z_Y be the only point in $\bigcap\{\mathrm{Cl}\, (p_X^Y)_*(U) : U \in u\}$, where $p_X^Y : Y \to X$ is a map from the system. The point z whose coordinates are z_Y just defined, Y running over the spaces of the system, is an element of Z, in virtue of Assertion 4. Clearly, $p_X(z) = x$.

The projection p_X is irreducible. In fact, let F be closed in Z and such that $F \neq Z$. Consider $W = Z - F$, open and non-empty. Since the system P is directed, there exists a space Y in the system and a non-empty open subset V of Y such that $p_Y^{-1}(V) \subset W$. Thus we have $\mathrm{Cl}\, p_Y(F) \subset Y - V$, and consequently $\mathrm{Cl}\, p_Y(F) \neq Y$. From the irreducibility of p_X^Y it follows that $\mathrm{Cl}\, p_X(F) \neq X$.

The projection p_X is an R-cover. In fact, let $u \in R$ and let us take an x in X such that $\{x\} = \bigcap\{\mathrm{Cl}\, U : U \in u\}$. Let z in Z be taken for x and for u as at the beginning of the proof, so that $p_X(z) = y$. For this z we get $z \in \bigcap\{\mathrm{Cl}\, (p_X)_*(U): U \in u\}$.

Now, using the Kuratowski-Zorn Lemma, we get the maximal elements in $\mathrm{Cov}_R X$.

Note that the maximal elements in $\mathrm{Cov}_R X$ are maximal in the whole $\mathrm{Cov}\, X$ if we neglect a difference in topologies modulo regularly open subsets (i.e., if we regard topologies as equivalent if the families of regularly open sets in these topologies are equal). To see this, let $p : E \to X$ be a maximal element in $\mathrm{Cov}_R X$. The space E is extremally disconnected, in virtue of the remark following Assertion 3. Then the announced fact follows from the assertion below (Błaszczyk [1]; for more usual version, see Gleason [3] and Flachsmeyer [2]):

6. *If E is extremally disconnected then each irreducible map onto E is a bijection which preserves regularly open subsets.*

There are many extremally disconnected covers for a given X, e.g. depending on the choice of R, $R \subset R(X)$. A partial ordering in the set $\mathrm{E\,Cov}\, X$, of extremally disconnected covers of X, is introduced if we set $p \leqq q$ iff there exists an h such that $q \circ h = p$ (notice the change of the order of composition in comparison to the ordering in the whole $\mathrm{Cov}\, X$). The map h which realizes the inequality is an "embedding" onto a dense subset, if we neglect the difference of topologies modulo regularly open subsets. We get the greatest extremally disconnected cover of X, if we take for R the whole $R(X)$; this is the Iliadis extremally disconnected cover as defined in [7], p. 31 (see also [6]).

References

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