THE MARTIN COMPACTIFICATION IN AXIOMATIC POTENTIAL THEORY

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1. Introduction. Let **B** denote an open ball in $\mathbf{R}^n$ $(n \geqq 2)$ of radius R and let S denote its boundary. The Poisson kernel $K(x, y)$ for B is defined by the formula

$$K(x, y) = \frac{R^{n-2}}{\alpha_n}\left[\frac{R^2 - \|x\|^2}{\|x - y\|^n}\right],$$

where α_n is the $(n - 1)$-dimensional surface area of S with $x \in B$ and $y \in S$.

It is well known that if f is a continuous real-valued function on S the function H_f defined by

$$H_f(x) = \int_S K(x, y) f(y)\, \mathrm{d}y,$$

with $\mathrm{d}y$ Lebesgue measure on S, solves the classical Dirichlet problem for B with boundary value f (cf. Helms [19]).

Since $x \to K(x, y)$ is harmonic, for all $y \in S$, it follows that for each positive Radon measure μ on S the function H_μ defined by

$$H_\mu(x) = \int_S K(x, y)\, \mu(\mathrm{d}y)$$

is positive and harmonic. The converse is true (theorem of Herglotz) and even further the measure μ is uniquely defined by the corresponding harmonic function. Hence, there is an integral representation for the positive harmonic functions on B in terms of the functions $K(., y)$, with $y \in S$, and the set $\mathscr{M}^+(S)$ of positive Radon measures on S.

In 1941 R. S. Martin showed that a similar integral representation holds for any bounded domain D in $\boldsymbol{R}^n$ [25]. Specifically, he proved the following result.

Theorem (R. S. Martin). *Let $D \subset \boldsymbol{R}^n$ $(n \geqq 2)$ be a bounded domain. Then there exists a metrizable compactification $\hat{D}$ of D with the following properties:*

(1) *to each point $y \in \Delta = \hat{D} \setminus D$ there corresponds a non-zero positive harmonic function $K(., y)$ such that $(x, y) \to K(x, y)$ is continuous on $D \times \Delta$;*

(2) *if H_μ is defined by*

$$H_\mu(x) = \int K(x, y)\, \mu(\mathrm{d}y)$$

with $\mu \in \mathcal{M}^+(\Delta)$, the correspondence $\mu \to H_\mu$ maps $\mathcal{M}^+(\Delta)$ onto the cone $\mathcal{H}^+(D)$ of positive harmonic functions on D; and

(3) *there is a G_δ-set $\Delta_1 \subset \Delta$ such that the map $\mu \to H_\mu$ is a bijection of $\{\mu \in \mathcal{M}^+(\Delta) \mid \mu(\Delta \setminus \Delta_1) = 0\}$ onto $\mathcal{H}^+(D)$.*

Definition. The compact metric space $\tilde{D}$ is called the *Martin compactification of D* and the ideal boundary $\Delta = \tilde{D} \setminus D$ is called the *Martin boundary* of D. The G_δ-set Δ_1 is called the set of *minimal points of Δ*.

The purpose of this paper is to discuss certain aspects of subsequent research dealing with a description of the Martin compactification and to mention some open problems.

2. Axiomatic potential theory. Martin's original work has been extended to three different contexts: axiomatic potential theory; and probabilistic potential theory both discrete and non-discrete. The following is a very brief outline of Brelot's axiomatic theory of potential (cf. [4] 1960 and also [1] 1966, [11] 1971 for the more general theories).

Let E be a connected, locally connected, locally compact, non-compact space with a countable base. Denote by $\mathcal{H}$ a sheaf on E.

Axiom 1. $\mathcal{H}$ is a sheaf of vector spaces of continuous real-valued functions on E.

Definition 2.1. A relatively compact open set $W \subset E$ will be said to be *$\mathcal{H}$-regular* if to each $f \in \mathcal{C}(\partial W)$ there corresponds a unique function $H_f \in \mathcal{H}(W)$ such that:

(1) $f \cup H_f \in \mathcal{C}(\overline{W})$; and

(2) $(f \geqq 0)$ implies $(H_f \geqq 0)$.

Axiom 2. E has a base of $\mathcal{H}$-regular sets.

Axiom 3. Let $W \subset E$ be open and connected. If $(h_n)_n \subset \mathcal{H}(W)$ is increasing then either $\sup_n h_n \in \mathcal{H}(W)$ or it is identically equal to $+\infty$.

Definition 2.2. A sheaf $\mathcal{H}$ on E that satisfies Axioms 1, 2 and 3 will be called a (Brelot) *harmonic sheaf.*

To each regular set W there correspond the *harmonic measures* μ_x^W, $x \in W$, defined by

$$\langle \mu_x^W, \varphi \rangle = H_f(x)\,,$$

where $f = \varphi \mid \partial W$ and $\varphi \in \mathscr{C}_c(E)$. It can be seen that the mapping $x \to \langle \mu_x^W, f\rangle$ is Borel-measurable for any non-negative Borel function f. Consequently, the operator H_W is a kernel (cf. [27] for a definition) where

$$H_W f(x) \text{ equals } f(x) \text{ if } x \notin W \text{ and } \langle \mu_x^W, f\rangle$$

if $x \in W, f \geqq 0$ Borel-measurable.

Definition 2.3. A lower semi-continuous function $u : E \to (-\infty, +\infty]$ is said to be $(\mathscr{H})$-*hyperharmonic* (resp. $(\mathscr{H})$-*hyperharmonic on an open set* U) if $H_W u \leqq u$ for each regular set W (resp. for each regular set W with $\overline{W} \subset U$). A hyperharmonic function is said to be *superharmonic* if it is finite on a dense set. A continuous function h is said to be *harmonic on an open set* U if h and $-h$ are both hyperharmonic on U. A superharmonic function u is called a *potential* if $h \leqq u$ and h harmonic (on E) implies $h \leqq 0$, and $u \geqq 0$.

Each non-negative superharmonic function u has a unique Riesz decomposition $u = p + h$, where p is a potential and h is harmonic.

The *support* of a hyperharmonic function u is defined to be the complement of the largest open set on which it is harmonic.

Hypothesis I. There exists a positive potential.

Assuming this hypothesis one can prove that for each $y \in E$ there exists a potential with support $\{y\}$. It is natural to ask if such potentials are unique, up to a constant. While this is not true in general (see Constantinescu and Cornea [10] 1968 for a counter example) in a very large number of cases this is in fact so.

Hypothesis II. (The hypothesis of proportionality.) For each $y \in E$, if p_1 and p_2 are potentials with support $\{y\}$ there exists $\lambda > 0$ with $p_1 = \lambda p_2$.

Definition 2.4. A lower semi-continuous function $G : E \times E \to [0, +\infty]$ is called a *Green function for* $\mathscr{H}$ if

(1) G is continuous off the diagonal; and
(2) for each $y \in E$, $x \to G(x, y)$ is a potential of support $\{y\}$.

In her thesis Mme Hervé proved that Hypotheses I and II imply the existence of Green functions for $\mathscr{H}$ ([17] 1962, Proposition 18.1).

3. Examples of harmonic sheaves. Let E be an open set in $\boldsymbol{R}^n$ $(n \geqq 2)$ and let $\mathscr{H}$ be the sheaf of $\mathscr{C}^2$-functions h for which $Lh = 0$ where L has one of the following forms:

(1) $L = \Delta$ (classical potential theory);

(2) $L = \sum_{i,j} a_{ij} \dfrac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \dfrac{\partial}{\partial x_i} + c$

where i) b_i, c and a_{ij} are locally Lipschitz, and

ii) (a_{ij}) is a symmetric matrix whose associated quadratic form is positive definite [17] 1962;

(3) $L = \sum_i \frac{\partial}{\partial x_i}\left(\sum_j a_{ij}\frac{\partial}{\partial x_j}\right)$ with the a_{ij} Lebesgue measurable, (a_{ij}) symmetric and $\sum_{i,j} a_{ij}\xi_i\xi_j \geqq \varepsilon(\sum_i \xi_i^2)$ for some $\varepsilon > 0$, uniformly on E [18] 1964.

Then $\mathscr{H}$ is a harmonic sheaf that satisfies hypotheses I and II.

Information on further refinements of the example (2) is given in the survey article [7] 1970, where Bony's important work [2] 1967 on determination of an elliptic operator by a harmonic sheaf is also discussed.

If E is a $\mathscr{C}^2$-manifold then any elliptic operator L on E that satisfies (2) or (3) locally defines a harmonic sheaf. In particular if E is a Riemannian manifold, the Laplace-Beltrami operator defines a harmonic sheaf on E.

Finally, the harmonic functions on a Riemann surface E form a harmonic sheaf on E.

4. The Martin compactification. Let $(K_\alpha)_{\alpha\in I}$ be a family of continuous functions $K_\alpha : E \setminus D_\alpha \to \overline{R}$, $D_\alpha \subset E$ compact. Then, a very slight modification of arguments given by Constantinescu and Cornea in [9] 1963 shows that there exists a unique compactification $\tilde{E}$ of E with the following properties:

(1) for each α, K_α extends continuously to $\tilde{E} \setminus D_\alpha$; and

(2) the extended functions separate the points of $\tilde{E} \setminus E$ (see [31] 1970).

This compactification will be said to be *defined by* $(K_\alpha)_{\alpha\in I}$ (it is a Q-compactification in the terminology of [9]) and can be described in various equivalent ways (uniform structures, proximity spaces, etc.).

Let $x_0 \in E$ and let G be a Green function for $\mathscr{H}$. Define $K(x, y)$ to be 1 if $x = x_0 = y$ and to be $G(x, y)/G(x_0, y)$ otherwise. The *Martin compactification of E determined by* $\mathscr{H}$ is defined to be the compactification defined by the family $(K_x^*)_{x\in E}$, where $K_x^*(y) = K(x, y)$. It is clearly independent of the choice of Green function in view of Hypothesis II and can be shown to be independent of x_0 (with a suitable topology on the cone of positive superharmonic functions it can be seen to be the closure of the set of extreme points of a compact base for this cone). Further, it is metrizable as the argument given in [5] 1971 (p. 112) shows.

In the case of the harmonic sheaf $\mathscr{H}$ defined by a suitable elliptic operator L, the Martin compactification for L defined by Shur in [29] 1962 coincides with the compactification defined by the sheaf.

Let $y \in \Delta = \tilde{E} \setminus E$. Then there is a sequence $(y_n)_n \subset E$ with $y = \lim_n y_n$, i.e., for each $x \in E$

$$\lim_{n\to\infty} K(x, y_n) = K_y(x)$$

exists. If $x_1 \in E$ and W is a connected open set containing x_0 and x_1 the functions $K(.,y_n)$ are harmonic on W for sufficiently large n. They all take value 1 at x_0 and since Harnack type inequalities hold in this setting, the function K_y is locally a uniform limit of harmonic functions and hence harmonic.

The function K is extended to $E \times \tilde{E}$ by using the functions K_y, $y \in \Delta$ and the desired integral representation follows by means of the arguments of Martin (c.f. [5] p. 113 for an exposition). Hence, Martin's theorem holds for any Brelot harmonic sheaf that satisfies Hypotheses I and II.

5. The relationship between $\tilde{W}$ and $\bar{W}$. Let $W \subset E$ be relatively compact and connected. The sheaf $\mathscr{H}$ on E induces a harmonic sheaf on W which satisfies Hypotheses I and II whenever $\mathscr{H}$ does. Hence W has a Martin compactification $\tilde{W}$. If W is an open ball in R^n and $\mathscr{H}$ is the sheaf defined by Laplace's operator it follows from the results stated in the introduction that $\tilde{W} = \bar{W}$ (the closure of W). It is therefore natural to ask in the general case what relationship holds between $\tilde{W}$ and $\bar{W}$?

In 1933 de la Vallée Poussin [24] obtained a Poisson type integral representation for a domain W in R^3 of finite connection bounded by a finite number of "sufficiently regular" surfaces (at each point of the boundary a tangent plane exists and the angle $\theta(x_1, x_2)$ between the normals at x_1 and x_2 tends to zero as $x_1 - x_2$ tends to zero) and a finite number of closed sets of capacity zero. The boundary points are identified with harmonic functions and as in the case of an open ball there is a bijection between $\mathscr{M}^+(\partial W)$ and $\mathscr{H}^+(W)$. When no exceptional sets of capacity zero exist in the boundary then W is a C^1-domain and so $\tilde{W} = \bar{W}$ by [20] 1970. It would be of interest to have a good modern exposition of this elegant paper [24]. For a very readable account of it and of Martin's work see Deny [12] 1947.

In 1970 R. A. Hunt and R. L. Wheeden in [20] published a proof of the fact that $\tilde{W} = \bar{W}$ (relative once again to Laplace's operator) if W is a bounded Lipschitz domain in R^n ($n \geqq 2$).

If W is a simply connected plane domain with at least two boundary points the Riemann mapping theorem states that there is a conformal map $\Phi : W \to (|z| < 1)$. The definition of the prime ends of W given by Carathéodory in [6] 1913 implies that they are the inverse images under Φ of the traces on $(|z| < 1)$ of the neighbourhood filters in $(|z| \leqq 1)$ of the points of the circle $(|z| = 1)$. Hence, the prime ends are the points of the Martin boundary for W. For W equal to the open square $(\max\{|x|, |y|\} < 1)$ minus the lines A_n, where $A_n = \{(x, y) \mid y \leqq 0,\ x = 1-2^{-n}\}$ it is well known from the theory of prime ends (cf. [6]) that $\tilde{W}$ and $\bar{W}$ are not comparable compactifications (i.e., there is no continuous map from one onto the other which is the identity map on W).

If $A \subset E$ is a closed set of capacity zero then $E \setminus A$ is connected and dense in E. Since the cone of positive harmonic functions on $E \setminus A$ can be identified with the cone of superharmonic functions on E whose support lies in A, it then follows that $\widetilde{E \setminus A} = \tilde{E}$ with Martin boundary equal to $\Delta \cup A$, where $\Delta = \tilde{E} \setminus E$. Further, each point of A is minimal. Consequently, if $\tilde{W} = \overline{W}$ and $A \subset W$ is closed and of capacity zero then $\widetilde{W \setminus A} = \overline{W \setminus A}$.

In the case of a general harmonic sheaf nothing is known about the relation between $\tilde{W}$ and $\overline{W}$. For example, it is not known if E has a base of relatively compact domains W for which $\tilde{W} = \overline{W}$. More specifically, if L is an elliptic operator on an open set $U \subset \boldsymbol{R}^n$ (of the type that defines a Brelot sheaf) and if W is even a very "regular" relatively compact domain with $\overline{W} \subset U$ nothing is known about the relation between $\tilde{W}$ and $\overline{W}$. Clearly, if everything is "sufficiently regular" one expects to have $\tilde{W} = \overline{W}$.

The same question can be considered for $\overline{W}$ a compact submanifold with boundary of a Riemannian manifold with interior W, the sheaf here being defined by the Laplace-Beltrami operator. Again one expects that $\tilde{W} = \overline{W}$. Hopefully, a solution of the local problem will solve this global one.

6. The existence of non-minimal points. In all the examples so far discussed $\Delta = \Delta_1$. Martin [25] constructed an example of a bounded domain W in $\boldsymbol{R}^3$ for which $\Delta \neq \Delta_1$ (it even satisfied the condition used by de la Vallée Poussin in [24]). Ikegami [21] 1967 proved that in the case of the Laplace operator (in fact for the somewhat more general case of a Green space) that $\Delta \neq \Delta_1$ implies $\Delta \setminus \Delta_1$ is infinite. Toda [32] 1967 then showed that it is even uncountable. These results have not been proved for a general harmonic sheaf. As a side comment it is noted that while in probabilistic potential theory the minimal points are the points to which the random particle almost surely converges as time goes to infinity, no probabilistic interpretation has been given of the non-minimal points.

In [25], Martin asked whether in general Δ_1 is dense in Δ. This question was resolved in striking fashion by Constantinescu and Cornea in [8] 1958. They showed that to each integer $n \geqq 1$ there corresponds a hyperbolic Riemann surface E_n whose Martin boundary is connected and contains exactly n minimal points. It would be interesting to know what part of the standard $(n - 1)$-simplex corresponds to the Martin boundary of E_n. One can ask for what compact convex sets K do there exist Brelot sheaves such that the corresponding set Δ_1 can be identified with $\mathscr{E}(K)$. In probabilistic potential theory the following articles provide results in this direction, [28] 1966 and [34] 1960.

These examples of Constantinescu and Cornea raise the question: is there a topological property of E which implies $\Delta = \Delta_1$ or more generally $\overline{\Delta}_1 = \Delta$? Note that for simply connected plane domains $\Delta = \Delta_1$ and that in the examples E_n the homology groups of $E_n \setminus A$, for any compact $A \subset E_n$, are presumably infinitely generated. For example, if E is contractible to a point is $\overline{\Delta}_1 = \Delta$?

7. Entrance and exit boundaries. In [9] Constantinescu and Cornea proved that the Martin compactification of a hyperbolic Riemann surface E is the compactification defined by the family of continuous functions g of the form

$$g = H_U f / H_U 1 ,$$

where f is continuous and bounded on E, $\complement U$ is compact and outer regular and H_U is the kernel that solves the Dirichlet problem for U. As a consequence it was proved in [9] that $\tilde{E} \geqq \hat{E}$, where $\hat{E}$ is the Stoilow compactification of E (i.e., the one defined by those continuous bounded functions f that are constant on the components of the complement of some compact set $A = A(f) \subset E$) and "$\geqq$" means that there is a continuous map from $\tilde{E}$ onto $\hat{E}$ which is the identity on E. This result is equivalent to having a bijection between the connected components of Δ and the ends of E.

While for a general sheaf H the above description of the Martin compactification is no longer valid, it is still true that $\tilde{E} \geqq \hat{E}$ [31] 1970. In the general setting this is a consequence of the fact that if $A \subset E$ is compact then $\tilde{E} \setminus A$ can be canonically embedded in the topological sum of the Martin compactifications of the connected components of $E \setminus A$.

The family of functions defined at the beginning of this section defines a compactification, *the entrance compactification* of E [30] 1969 (by analogy with the probabilistic entrance compactification defined by Doob in [13] 1959). It is not known, for a Brelot sheaf satisfying Hypotheses I and II whether the entrance and the Martin (or exit) compactifications coincide.

When the Green function G defines an adjoint sheaf $\mathscr{H}^*$ which satisfies the hypothesis of proportionality (Hypothesis II) the entrance compactification of E is the Martin compactification of E determined by $\mathscr{H}^*$ [30] 1969. If for a sufficiently regular elliptic operator L defined on a neighbourhood of a sufficiently regular relatively compact domain $W \subset R^n$ $(n \geqq 2)$ it follows that $\tilde{W} = \overline{W}$, then in this case entrance and exit compactifications will coincide (the sheaf $\mathscr{H}^*$ will be defined by the formal adjoint L^* of L).

B. Walsh in [33] 1969 has defined the notion of a *normal structure* $\mathscr{L}$ associated with a Brelot sheaf $\mathscr{H}$. It is a family $(N_A)_A$ of kernels indexed by a family of compact sets A whose interiors cover E such that the following conditions are satisfied:

N_1) f continuous implies $N_A f$ continuous;

N_2) $A_1 \subset A_2$ implies $N_{A_2} \circ N_{A_1} = N_{A_1}$; and

N_3) f bounded Borel implies $N_A f$ is harmonic on $\complement A$.

A normal structure $\mathscr{L}$ defines a subsheaf $\mathscr{H}^{\mathscr{L}}$ of $\mathscr{H}$ by requiring $N_A h = h, \forall A \subset \complement U$ if $h \in \mathscr{H}^{\mathscr{L}}(U)$ and so regulates the "behaviour at infinity" of the harmonic functions.

To each normal structure one can associate an entrance (and exit or Martin) compactification. Presumably, there are probabilistic interpretations of these compactifications.

Dynkin in [16] 1965 considered an abstract type of "boundary condition" which may have some relation to the notion of a *normal structure.*

8. Stability of the compactification. The Martin compactification of a locally compact space E with countable base depends a priori on the Brelot sheaf $\mathscr{H}$. In [31] 1970 the question of stability was considered and the following result proved.

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Brelot sheaves on a locally compact space such that both of them satisfy Hypotheses I and II. If there exists a compact set $A \subset E$ such that the sheaves agree on $E \setminus A$ then the corresponding Martin compactifications coincide and the corresponding sets of minimal points coincide.

For an elliptic operator L it would be of interest to know what perturbation of the coefficients, other than an "arbitrary" one on a compact set, leaves the Martin compactification invariant.

The stability question is not trivial as the following example shows. Consider E an open ball B of radius 1 in R^3 minus the ray A equal to $(x = y = 0,\ z \geqq 0)$. Since A has capacity zero the Martin compactification of E corresponding to Laplace's operator is $\bar{B}$ and the Martin boundary is $\partial B \cup A$. By analogy with the prime ends of a slit disk it is natural to try to construct prime ends for E that ramify A, replacing each point except $(0, 0, 0)$ by a circle. This can be done by using a suitable elliptic operator. Specifically, let L be the operator E obtained by transporting Laplace's operator on X to E by means of a diffeomorphism, where X is the open ball B minus the cone $(x^2 + y^2 = z^2,\ z \geqq 0)$. The Martin boundary of X is homeomorphic to a 2-sphere (by [20]) and so the Martin compactification of E determined by L ramifies A as desired. Hence this compactification differs from the one defined by Laplace's operator.

This example serves also to emphasize a point made by other authors. Namely, the theory of prime ends is really a study of the Martin boundary. Furthermore, the work done by Kaufmann (cf. [23] 1930) and later Mazurkiewicz [26] 1945 generalizing the theory of prime ends to higher dimensions defined objects which seem to be less maniable than the Martin compactification. Brelot in [3] 1946 showed (for the Laplacian) that in general Martin's and Mazurkiewicz' compactifications are not comparable. It could be that Mazurkiewicz' compactification can always be obtained by using a suitable sheaf.

9. The description of the boundary associated with an elliptic operator L. According to Dynkin in [15] 1963 "the set of minimal nonnegative solutions of the elliptic differential equation $Lf = 0$ has received little attention and provides many interesting unsolved problems. We will state two such problems.

The first problem is: under what restrictions on the differential operator L can be the set of minimal functions be provided with the structure of a smooth manifold such that $K_y(x)$ is a smooth function of y and x $(y \in \Delta)$?

The second problem concerns the connections between the geometry of a complete Riemannian manifold and the Martin boundary of this manifold", in particular the relation of the dimension of the boundary to the dimension of the manifold.

Dynkin himself in [14] 1961 and Karpelevich in [22] 1963 have shown that for certain homogeneous spaces of non-positive curvature the set of minimal functions can be explicitly described and has dimension one less than that of the manifold.

If an open Riemannian manifold E is isometrically embedded onto the interior of a compact Riemannian manifold with boundary then it is to be expected that, under certain conditions, this boundary is the Martin boundary of E. In other words, the Martin boundary should be the "natural" boundary to add to such a manifold.

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