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BANACH ALGEBRAS WITH INVOLUTION

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In this lecture we intend to present a report about some recent results in the theory of Banach algebras with involution. They are all based on an inequality in hermitian algebras discovered about a year ago which has brought considerable improvements in the theory of hermitian algebras as well as in abstract characterizations of C^*-algebras.

Let us begin by recalling some definitions and known results. Given a Hilbert space H, we denote by $B(H)$ the set of all bounded linear operators on H. It becomes a Banach algebra if we equip it with the operator norm and the usual algebraic structure. Also, it carries a natural involution, the mapping which assigns to each $T \in B(H)$ its adjoint T^*. A C^*-algebra is defined as a closed selfadjoint subalgebra of $B(H)$ which contains the identity operator I. The starting point of the theory of C^*-algebras is the 1943 paper of I. M. Gelfand and M. A. Naimark [3] where the first abstract characterization of a C^*-algebra was given. Before stating their result and describing its further developments let us mention some important properties of C^*-algebras which admit an abstract formulation and which might be expected to be characteristic for C^*-algebras.

First of all, let us mention some obvious relations between the involution and the norm. First of all, the involution is isometric

$$|T^*| = |T| \quad \text{for all} \quad T \in A\,.$$

Also, it is easy to see that $|T^*T| = |T|^2$, or, using the isometry of the involution

$$|T^*T| = |T^*|\,|T| \quad \text{for all} \quad T \in A\,.$$

Let us pass now to some less superficial properties which do not involve explicitly the norm.

The fact that the elements of a C^*-algebra A are operators on a given concrete space enables us to formulate certain properties of the operators in terms of vectors of the underlying space. One example of such a formulation is the description of an operator T based on the properties of its quadratic form (Tx, x). It is a well known fact that an operator T is selfadjoint if and only if its quadratic form assumes only real values for every $x \in H$. Also, the spectrum of a selfadjoint element $T \in A$ is real. (The spectrum of T in A is the same as the spectrum of T taken as an element of

$B(H)$.) An element $T \in A$ is said to be nonnegative if its quadratic form assumes nonnegative values for every $x \in H$. It follows from what has been said above that a nonnegative T is necessarily selfadjoint. Also, the spectrum of T is nonnegative. Given any operator $S \in A$, we can construct a nonnegative operator in A by taking the product S^*S. Indeed, it is easy to see that its quadratic form is nonnegative as follows

$$(S^*Sx, x) = (Sx, Sx) = |Sx|^2 \geqq 0 .$$

Furthermore, every nonnegative operator in A may be obtained in this manner. This follows from the following theorem which is not difficult to prove.

Let A be a C^*-algebra and $T \in A$. Then the following conditions are equivalent:

(1) the quadratic form (Tx, x) is nonnegative,
(2) T is selfadjoint and its spectrum is nonnegative,
(3) $T = B^2$ for a suitable selfadjoint $B \in A$,
(4) $T = S^*S$ for a suitable $S \in A$.

In particular, for every $T \in A$ the spectrum in A of $I + T^*T$ does not contain zero and hence $I + T^*T$ has an inverse in A.

Now there are essentially two possibilities of stating abstract analogues of these properties. The first one (the one which we shall adopt) consists in imposing conditions describing spectral properties of selfadjoint elements or of elements of the form T^*T. The other (inaugurated by Vidav [15]) consists in replacing the quadratic form (which requires the vectors of the underlying space) by other notions defined intrinsically in terms of the algebra itself.

Let us state now the classical result of Gelfand and Naimark.

Theorem 1. *Let A be a (complex unital) Banach algebra with an involution which satisfies the following conditions:*

1° $(x + y)^* = x^* + y^*$ *for all* $x, y \in A$,
2° $(xy)^* = y^*x^*$ *for all* $x, y \in A$,
3° $(\alpha x)^* = \alpha^* x^*$ *for all* $x \in A$ *and all complex* α,
4° $|x^*x| = |x^*|\,|x|$ *for all* $x \in A$,
5° $|x^*| = |x|$ *for each* $x \in A$,
6° $(1 + x^*x)^{-1}$ *exists for each* $x \in A$.

Then A is isometrically $*$-isomorphic to a closed selfadjoint subalgebra of $B(H)$ for a suitable Hilbert space H.

We observe that the first three conditions describe the standard algebraic properties of the involution: the $*$-operation is to be a conjugate-linear anti-isomorphism of period two. The fourth condition, $|x^*x| = |x^*|\,|x|$, relates the involution to the metric structure. The last two conditions are of a different character; their introduction was necessitated by the method of proof and the authors themselves conjectured that these two conditions are consequences of the first four. This is indeed the case but much work was necessary to develop methods which yield the result without the last two assumptions.

The fifth condition, $|x^*| = |x|$, asserts that the involution is isometric, in particular continuous. The sixth asserts that elements of the form $1 + x^*x$ have an inverse, in other words, that the spectrum of an element of the form x^*x does not contain any negative numbers.

Let us examine now more closely what sort of problems the suppression of the last two conditions introduces. First of all, without condition five, we do not have continuity of the involution; it follows that delicate methods have to be developed which avoid using the continuity of the involution. Extreme care is indicated since we can no longer use many facts which are obvious in the case of operator algebras.

To mention one: since we do not have continuity of the *-operation, we cannot assert that the set of all selfadjoint elements (the kernel of the mapping $f(x) = x - x^*$) is closed.

It was only in 1960 that the fact that conditions five and six are consequences of the first four conditions or, in other words, that – apart from the standard conditions concerning the involution – the condition $|x^*x| = |x^*|\,|x|$ alone is sufficient for a Banach algebra to be isomorphic to an operator algebra was established by I. M. Glimm and R. V. Kadison [5].

It is customary to use the abbreviation B^*-algebra for Banach algebra which satisfies the first four conditions of the preceding theorem. The result of Glimm and Kadison can thus be formulated shortly as follows: every B^*-algebra is a C^*-algebra. (It should be mentioned at this point that there is a similar but much stronger condition which appears sometimes instead of condition 4°. This condition, $|xx^*| = |x|^2$, implies immediately the isometry of the involution – indeed, it is equivalent to conditions 4° and 5° taken together. The proof that a Banach algebra satisfying 1°, 2°, 3° and $|x^*x| = |x|^2$ is C^* can, accordingly, use the isometry of the involution and can be established essentially by the methods of Gelfand and Naimark.)

To return to condition six: we observe that it has a purely algebraic character. Let us state, at this point, two more algebraic conditions which will play an important rôle later:

7° *if $x \in A$ is selfadjoint then $\sigma(x)$ is real,*

8° *for every $x \in A$ the spectrum $\sigma(x^*x)$ is nonnegative.*

Also, some more terminology: a Banach algebra satisfying 1°, 2° and 3° will be called a Banach algebra with involution. Let us stress the fact that continuity of the involution is not required. A Banach algebra with involution is said to be hermitian if it satisfies condition 7°. We observe first that 8° is a stronger form of 6°; we shall see later that they are equivalent. Furthermore, it is not difficult to see that, in any Banach algebra with involution, condition 8° implies condition 7°. Indeed, assume 8° and consider a selfadjoint $h \in A$ and a $\lambda \in \sigma(h)$. It follows that $\lambda^2 \in (\sigma(h))^2 =$

$= \sigma(h^2) = \sigma(h^*h)$; the last set consists of nonnegative numbers only by our condition, so that λ has to be real. The question whether, conversely, hermitian algebras also satisfy condition 8°, remained open until quite recently.

Algebraic investigations, notably by I. Kaplansky [7], have put into evidence an intimate relation between condition six and eight and the fact that the sum of two selfadjoint elements with nonnegative spectra is again an element whose spectrum is nonnegative.

Let us try now to describe what we take to be the decisive steps in the further development.

First of all, it was necessary to investigate the possibility of extracting selfadjoint square roots from selfadjoint elements in algebras where continuity of the involution is not assumed. A lemma of J. Ford [2] asserts the existence of a selfadjoint u such that $u^2 = h$ provided h is a selfadjoint element whose spectrum lies in the open right half-plane. This lemma, together with methods due to Kaplansky, has enabled S. Shirali and J. Ford [14] to prove that, in a hermitian algebra, elements of the form x^*x have nonnegative spectra. (We have seen already that an algebra with this property must be hermitian.)

The second important contribution is the theorem of B. Russo and H. A. Dye [13] according to which the unit cell of a C^*-algebra coincides with the closed convex hull of its unitary elements. This result has been greatly improved and exploited by T. W. Palmer [8]. Using a transformation introduced by V. P. Potapov [10], L. A. Harris [6] has been able to develop a very simple non-spatial method of proof.

In a recent paper [11] the author showed that, in a Banach algebra with hermitian involution, the spectral radius of any element x and the spectral radius of x^*x are related by the following fundamental inequality

$$|x|_\sigma^2 \leqq |x^*x|_\sigma .$$

It was this result which made it possible to recognize fully the importance of the function $p(x) = |x^*x|_\sigma^{1/2}$, the square root of the spectral radius of the element x^*x; it will play a rôle not unlike the modulus of a complex number.

Once the fundamental inequality is established it is fairly easy to prove the following theorem.

Theorem 2. *Let A be a Banach algebra with hermitian involution. Then $|x|_\sigma \leqq p(x)$ for all $x \in A$. The function p is an algebra pseudonorm on A which coincides with $|x|_\sigma$ on normal elements. The kernel $p^{-1}(0)$ coincides with the radical of A.*

It should be mentioned that the fundamental inequality is an algebraic analogue of the B^*-condition $|x^*x| = |x^*|\,|x|$. To see that, let us note first that the B^*-condition is equivalent to the inequality $|x^*|\,|x| \leqq |x^*x|$. This is an immediate consequence of the submultiplicativity of the norm. If we replace norms by spectral radii in the preceding inequality, we obtain – using the obvious fact that $|x^*|_\sigma = |x|_\sigma$ for each $x \in A$ – the inequality $|x|_\sigma^2 \leqq |x^*x|_\sigma$, which is, indeed, the fundamental inequality. It is possible also to show that this inequality is actually equivalent to the assumption that the involution is hermitian. This analogy between hermitian algebras and B^*-algebras suggests that it may be possible to obtain most of the results known for B^*-algebras under the purely algebraic assumption that the algebra is hermitian. This is, indeed, the case. It is also possible to obtain considerable simplifications in the proof that B^* is C^* and in the theory of algebras which may be renormed to become B^*. In particular, the fact that p is an algebra pseudonorm in hermitian algebras plays an important rôle in the proofs. This fact has a deeper significance: indeed, p is an algebra pseudonorm if and only if the algebra is hermitian.

Also, methods using the fundamental inequality can be used to establish the equivalence between 7° and 8° in a simple manner. Before we state a theorem summarizing the recent state of our knowledge of hermitian algebras, let us define an algebra pseudonorm which will enable us to prove an analogy of the theorem of Russo and Dye for hermitian algebras. We denote by q the function defined on A as follows. Given $x \in A$, let $q(x)$ be the infimum of the sums $\sum \lambda_t$, $t \in K$, where K is a finite set, λ_t are positive numbers and $x = \sum \lambda_t u_t$, $t \in K$ for suitable unitary elements u_t. (An element u is said to be unitary if $u^*u = uu^* = 1$.)

Theorem 3. *Let A be a Banach algebra with involution. Then the following conditions are equivalent*:

1 *the involution is hermitian,*

2 $|x|_\sigma \leqq p(x)$ *for every* $x \in A$,

3 $|x|_\sigma \leqq p(x)$ *for every* $x \in N(A)$,

4 $|x|_\sigma = p(x)$ *for every* $x \in N(A)$,

5 $|\frac{1}{2}(x^* + x)|_\sigma \leqq p(x)$ *for every* $x \in A$,

6 p *is subadditive,*

7 $|u|_\sigma = 1$ *for every unitary* $u \in A$,

8 $|u|_\sigma \leqq 1$ *for every unitary* $u \in A$,

9 $|u|_\sigma \leqq \beta$ *for every unitary* $u \in A$ *and some* β,

10 x^*x *has nonnegative spectrum for every* $x \in A$,

11 *the real part of every* $\lambda \in \sigma(x^*x)$ *is nonnegative,*

12 $\sigma(x^*x)$ *does not contain negative numbers,*

13 $1 + x^*x$ *is invertible for every* $x \in A$,

14 $p(x) = q(x)$ *for every* $x \in A$.

The proof is based on the fundamental inequality and is contained in the paper [18] submitted for publication. Let us mention some of the implications which seem to be particularly interesting. The fundamental inequality is equivalent to the involution being hermitian. Also, unitary elements have bounded spectral radii if and only if the algebra is hermitian. We obtain also the equivalence of our former conditions 6°, 7° and 8°. The equivalence of 1 and 6 (of the present theorem) shows that hermitian algebras generate their own metric structure by means of p and that hermitian algebras form the natural boundary within which the function p has nice properties. If the algebra is not hermitian then p will not be subadditive. Also, the equivalence of 1 and 14 shows that hermitian algebras (and only these) share with C^*-algebras the important property that the unit ball of p is the closed convex hull of the unitary elements, another purely algebraic description of the natural pseudonorm p.

Let us mention another result [11] which illustrates the intimate relation in hermitian algebras between the algebraic structure and the topology generated by p.

Theorem 4. *Let A be a Banach algebra with hermitian involution. Let f be a linear form on A such that $f(1) = 1$. Then the following conditions are equivalent:*

1 $f(x^*x) \geqq 0$ *for every* $x \in A$,

2 $f(x) \leqq p(x)$ *for every* $x \in A$.

If we denote by $S(A)$ the set of all linear forms on A which possess these properties then $p(x) = \sup \{f(x^*x)^{1/2}; f \in S(A)\}$.

This result shows that the algebraic condition of positivity may be replaced by the topological condition of continuity with respect to p.

Also, the fundamental inequality together with a simple closed graph argument show that, in a hermitian algebra, p is continuous. Together with the fact that $p^{-1}(0)$ is the radical and the above theorem this gives a sufficient number of continuous representations (modulo the radical) in Hilbert space.

To show at least one of the applications of the theory let us prove a sharpening of the theorem that $B^* = C^*$.

Theorem 5. *Let A be a Banach algebra with involution which satisfies $|x^*x| = |x^*|\,|x|$ for each normal element $x \in A$. Then A is isometrically $*$-isomorphic to a C^*-algebra.*

Proof. Let $x \in A$ be normal. For every natural number n

$$|(x^*x)^n| = |(x^*)^n x^n| = |x^{*n}|\,|x^n|\,.$$

Hence $|x^*x|_\sigma = |x^*|_\sigma\,|x|_\sigma$ and A is hermitian by Theorem 3. Since $|h^2| = |h|^2$ for selfadjoint $h \in A$, we have $|h| = |h|_\sigma$. Hence $p(x)^2 = |x^*x|_\sigma = |x^*x| = |x^*|\,|x|$. Since unitary elements are normal, it follows from our assumption that $|u| = 1$ for every unitary $u \in A$, whence $|x| \leqq q(x)$ for each $x \in A$. The algebra being hermitian, we have $q(x) = p(x)$ for every x. It follows that $|x|^2 \leqq q(x)^2 = p(x)^2 = |x^*|\,|x|$ whence $|x^*| = |x|$ and $p(x) = |x|$. In particular, A is semisimple (the radical is $p^{-1}(0)$) and has, accordingly, a $*$-representation π on Hilbert space such that $|\pi(a)| = p(a)$ for $a \in A$. Hence $|a| = |\pi(a)|$ and π is an isometric $*$-isomorphism.

References

[1] *E. Berkson:* Some characterizations of C^*-algebras. Illinois J. Math. *10* (1966), 1—8.
[2] *J. W. M. Ford:* A square root lemma for Banach star-algebras. J. London Math. Soc. *42* (1967), 521—522.
[3] *I. M. Gelfand and M. A. Naimark:* On the imbedding of normed rings into the ring of operators on Hilbert space. Mat. Sb. *12* (1943), 197—213.
[4] *B. W. Glickfeld:* A metric characterization of $C(X)$ and its generalization to C^*-algebras. Illinois J. Math. *10* (1966), 547—556.
[5] *J. G. Glimm and R. V. Kadison:* Unitary operators in C^*-algebras. Pacific J. Math. *10* (1960), 547—556.
[6] *L. A. Harris:* Schwarz's lemma and the maximum principle in infinite dimensional spaces. Doct. Disser., Cornell Univ., 1969.
[7] *I. Kaplansky:* Symmetry of Banach algebras. Proc. Amer. Math. Soc. *3* (1952), 396—399.
[8] *T. W. Palmer:* Characterizations of C^*-algebras. Bull. Amer. Math. Soc. *74* (1968), 538—540.
[9] *T. W. Palmer:* The Gelfand-Naimark pseudo-norm on Banach *-algebras. J. London. Math. Soc. (2) *3* (1971), 59—66.
[10] *V. P. Potapov:* La structure multiplicative des fonctions matricielles J-contractives. Trudy Moskov. Mat. Obšč. *4* (1955), 125—236.
[11] *V. Pták:* On the spectral radius in Banach algebras with involution. Bull. London Math. Soc. *2* (1970), 327—334.
[12] *C. E. Rickart:* General theory of Banach algebras. Van Nostrand, Princeton, N. J., 1960.
[13] *B. Russo and H. A. Dye:* A note on unitary operators in C^*-algebras. Duke Math. J. *33* (1966), 413—416.
[14] *S. Shirali and J. W. M. Ford:* Symmetry in complex involutory Banach algebras. Duke Math. J. *37* (1970), 275—280.
[15] *I. Vidav:* Eine metrische Kennzeichnung der selbstandjungierten Operatoren. Math. Z. *66* (1956), 121—128.
[16] *B. Yood:* On axioms for B^*-algebras. Bull. Amer. Math. Soc. *76* (1970), 80—82.
[17] *D. A. Raikov:* To the theory of normed rings with involution. Dokl. Akad. Nauk SSSR *54* (1946), 387—390.
[18] *V. Pták:* Banach algebras with involution. Manuscripta Math. *6* (1972), 245—290.

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