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NORMAL AND CATEGORY MEASURES ON TOPOLOGICAL SPACES

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The general theory of topological spaces can be built without using real numbers. But many parts of the advanced topology have a substantial relation to real numbers. One of these relations turns up in the investigation of measures on topological spaces. Here we are interested in the case when topologically thin sets are thin in a measure theoretic sense too. Several authors have considered such questions with more or less restrictive assumptions on the underlying space. It is not immediately evident how significant these assumptions are. The purpose of this paper is to make some steps in an interesting program to clarify the relevance of a special topological structure for topological measures.

1. The space of regular Borel measures

Let X be an arbitrary topological space. A bounded Borel measure μ on X is a positive countably additive function on the Borel field of X (the least σ-field containing the open sets) with $\mu(X) < +\infty$. By regularity of a Borel measure μ we mean that for every Borel set B in X:

$$\mu(B) = \inf \{\mu(U) \mid U \text{ open in } X \text{ and } U \supset B\}.$$

A bounded signed Borel measure μ is called regular if its positive and negative parts μ^+ and μ^- are regular Borel measures.

The system $\mathscr{M}(X)$ of all bounded regular signed Borel measures with pointwise addition and scalar multiplication and pointwise ordering is a vector lattice. By definition

$$\|\mu\| := |\mu|(X) = \mu^+(X) + \mu^-(X).$$

$\mu \rightsquigarrow \|\mu\|$ is a norm for $\mathscr{M}(X)$ and makes it a Banach lattice.

Now a Banach lattice L is called a Kakutani-L-space if the norm has the additivity property for positive elements:

$$x \geqq 0, \quad y \geqq 0, \quad \|x + y\| = \|x\| + \|y\|.$$

Every Kakutani-L-space is an order complete vector lattice (cf. Day [2]). From the results mentioned in Dunford-Schwartz [4], Chap. III § 7, it can be seen that:

The system $\mathcal{M}(X)$ of all bounded signed regular Borel measures on an arbitrary topological space forms in a natural way a Kakutani-L-space.

2. The space of normal measures

The notion of a normal Borel measures is due to J. Dixmier [3] for the special case of extremally disconnected compact Hausdorff spaces. Some writers gave an extension of this notion to more general spaces (cf. Heider [8] for Boolean spaces, Knowles [11] for completely regular spaces, Hebert and Lacey [7] for compact Hausdorff spaces).

Definition. Let X be an arbitrary topological space. A bounded regular Borel measure $\mu \in \mathcal{M}^+(X)$ (the set of positive measures) is called a *normal Borel measure* if every nowhere dense Borel set has measure zero. Roughly speaking, a normal positive Borel measure is a measure for which topological null-sets are measure null-sets.

A *signed normal Borel measure* is a measure $\mu \in \mathcal{M}(X)$ such that μ^+, μ^- are normal Borel measures.

Theorem 1. (*The structure of the system of all normal Borel measures.*)

Suppose X is an arbitrary topological space. Then the system $\mathcal{N}(X)$ of all bounded signed normal Borel measures on X with respect to pointwise addition and scalar multiplication, and pointwise ordering is a vector lattice.

For the positive part μ^+ and the negative part μ^- of $\mu \in \mathcal{N}(X)$ it holds

$$\mu^+(B) = \sup\{\mu(A) \mid A \text{ Borel set with } A \subset B\}$$
$$\mu^-(B) = -\inf\{\mu(A) \mid A \text{ Borel set with } A \subset B\}.$$

By the total variation $\|\mu\| := |\mu|(X)$ ($|\mu| = \mu^+ + \mu^-$) of the measure $\mu \in \mathcal{N}(X)$ the vector lattice $\mathcal{N}(X)$ becomes a Banach lattice, this means complete with respect to the norm $\|.\|$ and the order and the norm structures are related through

$$|\mu| \leqq |\nu| \Rightarrow \|\mu\| \leqq \|\nu\|.$$

Moreover, the norm has the additivity property on $\mathcal{N}(X)$:

$$\mu \geqq 0, \quad \nu \geqq 0, \quad \|\mu + \nu\| = \|\mu\| + \|\nu\|.$$

Summarizing all these properties:

The space $\mathcal{N}(X)$ of all signed normal Borel measures on the topological space X is a Kakutani-L-space. $\mathcal{N}(X)$ is a convex lattice-subspace of $\mathcal{M}(X)$.

Remark. For the case of Boolean spaces X and normal Baire measures this theorem is stated in Heider [8], for the case of compact Hausdorff spaces and normal Borel measures it is remarked in Hebert-Lacey [7], p. 116.

Proof. The difference of two positive normal Borel measures μ_1, μ_2 must be normal, since $\mu = \mu_1 - \mu_2 = \mu^+ - \mu^-$ implies $\mu_1 \geqq \mu^+ \geqq 0$, $\mu_2 \geqq \mu^- \geqq 0$ and therefore μ^+ and μ^- are normal. Thus we have $\mathcal{N}(X)$ as a subspace of $\mathcal{M}(X)$. This subspace is a l-convex (a solid) subspace, namely for $\mu \in \mathcal{N}(X)$ and $\nu \in \mathcal{M}(X)$ $|\nu| \leqq |\mu|$ implies $\nu \in \mathcal{N}(X)$. To prove the norm completeness of the normal vector lattice $\mathcal{N}(X)$ we have only to follow the reasoning by which the norm completeness of $\mathcal{M}(X)$ is proved in Dunford-Schwartz [4], Chap. III § 7. There the equivalence of the total variation norm $\|.\|$ in $\mathcal{M}(X)$ and the supremum norm $\|\mu\|^{\bullet} := \sup \{|\mu(B)| \mid B \text{ Borel set in } X\}$ is shown. But for the norm $\| \ \|^{\bullet}$, $\mathcal{N}(X)$ is a closed subspace of $\mathcal{M}(X)$.

Examples. 1. *For every separable metrizable space X without isolated points there exists only the trivial normal Borel measure $\mu \equiv 0$. Thus $\mathcal{N}(X)$ is here the null space.*

Particularly, this can be established for a regular Borel measure $\mu(X) > 0$ with $\mu(\{x\}) = 0$ for each $x \in X$ by finding a dense open subset of arbitrary small measure using a cover of a countable dense subset (see for this question also Marczewski and Sikorski [12]).

2. *Let X be a T_1-space. Every bounded signed regular Borel measure on X is a normal Borel measure iff X is discrete.* ($\mathcal{M}(X) = \mathcal{N}(X)$.)

For this it is only needed to consider the point-measures (Dirac measures) δ_x, $x \in X$.

Recall that a *Freudenthal-unit* in a vector lattice is a positive element $\nu > 0$ such that $\inf(|x|, \nu) = 0$ implies $x = 0$.

The L-space $\mathcal{N}(X)$ for a discrete space X has a Freudenthal-unit iff card $X = \aleph_0$. *A positive measure on X is a Freudenthal-unit iff each point has strict positive measure. Of course $\mathcal{N}(X)$ $(=\mathcal{M}(X))$ is order isomorphic and isometric to the space $l_1(X)$ of absolute summable real "sequences" on X.*

3. Apart from its intrinsic interest, the relevance of normal Borel measures arises from the well known situation of representation of every Kakutani-L-space by a space of all normal measures on a suitable compact extremally disconnected Hausdorff space X (cf. Kelley and Namioka [15]). There the space X can be constructed as the structure space of the dual space L^* which is isometrically isomorphic to the space of all continuous real valued functions $\mathcal{C}(X)$ on a topologically uniquely determined compact Hausdorff space X. The canonical embedding of L in its second

dual L^{**} gives L, after identifying L^{**} with the space $\mathscr{M}(X)$ of all regular Borel measures on X, as the closed linear subspace $\mathscr{N}(X)$ of $\mathscr{M}(X)$.

According to this representation theorem every L-space $\mathscr{M}(X)$ of all bounded signed regular Borel measures on a topological space X is isometric isomorphic to the space $\mathscr{N}(Y)$ of all normal Borel measures on a space Y.

The space $\mathscr{M}(\alpha\mathbf{N})$ of all bounded signed regular Borel measures on the one-point compactification of the space $\mathbf{N}$ of natural numbers is isometric isomorphic to the space $\mathscr{N}(\mathbf{N})$ of all normal measures on $\mathbf{N}$ resp. isometric isomorphic to the space $\mathscr{N}(\beta\mathbf{N})$ of all normal Borel measures on the Stone-Čech compactification of the natural numbers.

By the way we can ask for which spaces X there is a "nice" correspondence to spaces Y such that $\mathscr{M}(X)$ is isometric isomorphic to $\mathscr{N}(Y)$.

3. Supports of normal measures

The support for $\mu \in \mathscr{M}^+(X)$ is defined as the following

$$\operatorname{supp} \mu := X \setminus \bigcup U \mid (U \text{ open in } X \text{ with } \mu(U) = 0).$$

For arbitrary elements $\mu \in \mathscr{M}(X)$, the support is understood to be the union of the supports of the positive part μ^+ and the negative part μ^-.

Definition. An arbitrary topological space X is said to have a *rich system of normal Borel measures* iff every nonvoid open subset contains the support of a non trivial normal Borel measure on X.

Remark. This concept applied to extremally disconnected Hausdorff spaces (sometimes called Stonian spaces) is equivalent to Dixmier's notion of hyperstonian spaces. According to Dixmier, in a hyperstonian space the union of supports of normal measures is dense in the space.

Theorem 2. *Suppose X is an arbitrary topological space. The support of a normal Borel measure on X is a regular-closed subset of X, which is a Baire space: If $\mu \in \mathscr{N}(X)$ then for $F := \operatorname{supp} \mu$, $F = \overline{\operatorname{Int} F}$, and every relatively open subset of F is not meager. Let X be a space with a rich system of normal Borel measures. Then every meager set is nowhere dense and X must be a Baire space in which every non empty open set contains a non empty regular closed set.*

Remark. For compact Hausdorff spaces X which are extremally disconnected (sometimes called Stonian spaces) this is the contents of two propositions by Dixmier [3]. The last part is a generalization of Theorem 3 in Oxtoby's paper [14]. Measures with the whole space as the support are considered there. Spaces that contain in every non empty open set a regular closed set are called quasi-regular by Oxtoby.

Proof. We can consider positive measures because the union of two regular closed sets is regular closed again. Let $\mu \in \mathcal{N}^+(X)$, $F := \operatorname{supp} \mu$. For the regular-closed kernel $\hat{F} := \overline{\operatorname{Int} F}$ of F, $F \setminus \hat{F}$ is a nowhere dense Borel set. Therefore $\mu(F \setminus \hat{F}) = 0$. Then $\hat{F}$ must be the support of μ, i.e. $F = \hat{F}$. μ restricted to its support is a normal Borel measure on its support. Then every nonvoid open subset of supp μ has strictly positive measure but meager open sets are null-sets with respect to normal measures. To prove the second part it is sufficient to consider Borel sets and show for meager Borel sets B, $\operatorname{Int} \bar{B} = \emptyset$. Each meager Borel set is for every normal measure a null set. Then $\operatorname{Int} B = \emptyset$, since for $\operatorname{Int} B \neq \emptyset$ there exists a positive Borel measure with $\mu(B) \geqq \mu(\operatorname{Int} B) > 0$. From $\bar{B} = (\bar{B} \setminus \operatorname{Int} B) \cup \operatorname{Int} B$ it follows $\mu(\bar{B}) = 0$ for each normal Borel measure. Thus we have

$$\operatorname{Int} \bar{B} = \emptyset .$$

Corollary. *Let X be a T_1-space separable in the sense of Fréchet (that means X has a countable dense subset) without isolated points. Then there is no nontrivial normal Borel measure on X.*

Proof. Let μ be a nontrivial normal Borel measure on X. Supp μ is regular closed in X. Then supp μ is a T_1-space separable in the sense of Fréchet without isolated points. Thus we may assume supp $\mu = X$. Then X is a space with a rich system of normal Borel measures. The countable dense set is meager but not nowhere dense. It follows $\mu \equiv 0$.

Remark. Dixmier [3] constructs an extremally disconnected compact Hausdorff space which is not hyperstonian in the sense of Dixmier but for which every meager set is nowhere dense.

Definition. Let X be an arbitrary topological space. A positive normal measure $\mu \in \mathcal{N}^+(X)$ is said to be a *category Borel measure* iff its support is the whole space X.

Remark. Oxtoby [14] called a positive bounded measure μ on the class of sets having the property of Baire (the σ-field generated by the open sets and the meager sets) a category measure for the topological space X if $\mu(A) = 0$ means A is meager. Every category measure is the completion of a Borel measure. It is easy to see that a category Borel measure in our sense gives a regular category measure of Oxtoby by completion and conversely a category measure of Oxtoby restricted to the class of Borel sets is a Borel category measure. In other words, a category Borel measure means a regular Borel measure such that the Borel null-sets are identical with the Borel meager sets (Borel sets of the first category).

Theorem 3. *Let X be a topological space with a rich system of normal Borel measures. Then the Kakutani-L-space $\mathcal{N}(X)$ of all normal Borel measures has the following properties:*

1. *The Freudenthal-units coincide with the Borel category measures on X.*
2. *There exist Freudenthal-units in $\mathcal{N}(X)$ iff the space X has the Souslin-property that every open disjoint family is countable.*

Remark. Without any assumption on the space X each Borel category measure in $\mathcal{N}(X)$ is a Freudenthal-unit in $\mathcal{N}(X)$, and the converse does not hold.

Proof. Let $\nu \in \mathcal{N}^+(X)$. If $\operatorname{supp} \nu \neq X$ then $X \setminus \operatorname{supp} \nu$ contains a support of a nontrivial normal Borel measure μ. Then $\inf(\nu, |\mu|) = 0$. Therefore Freudenthal-units must have X for its support. Conversely, for a category measure ν let $\inf(\nu, |\mu|) = 0$ with $\mu \in \mathcal{N}(X)$. If $U \neq \emptyset$ is open, then $\nu(U) > 0$. Therefore $|\mu|(U) = 0$ and $\operatorname{supp} \mu = \emptyset$. For a positive normal Borel measure μ with support X every disjoint open family must be countable.

Now let X be a space with a rich system of normal Borel measures and such that it has the Souslin-property. We consider a disjoint family in the vector lattice $\mathcal{N}(X)$. This means $(\mu_i)_{i\in I}$, $0 \neq \mu_i \in \mathcal{N}(X)$, with $\inf(|\mu_i|, |\mu_j|) = 0$ for $i \neq j$. Then since $\operatorname{Int}(\operatorname{supp} \mu_i)$ is a disjoint family of open sets, it must be a countable family, i.e. $\operatorname{card} I \leqq \aleph_0$. With the help of Zorn's Lemma we can find a maximal disjoint family of positive normal measures. This family is countable: $\mu_1, \mu_2, \ldots, \mu_n, \ldots$. Hence

$$\mu := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mu_n}{\|\mu_n\|}$$

is a normal measure. We have $\operatorname{supp} \mu = X$, since in the other case we find a positive normal measure ν with $\operatorname{supp} \nu \subset X \setminus \operatorname{supp} \mu$, which contradicts the maximality of the family $\mu_1, \mu_2, \ldots$.

4. Normal measures and their induced measures on the Boolean algebra of regular open sets

The regular open sets of a topological space X form (with respect to the inclusion as an ordering) a complete Boolean algebra. This Boolean algebra is isomorphic to the Boolean algebra of regular closed sets.

Definition. Let $\mathbf{B}$ be a complete Boolean algebra. A positive additive function $\mu : \mathbf{B} \to \mathbf{R}$ is said to be *σ-additive* if for every countable disjoint family $(a_n)_{n\in\mathbf{N}}$ of $\mathbf{B}$

$$\mu(\sup a_n) = \sum_{n=1}^{\infty} \mu(a_n),$$

completely additive if for every arbitrary disjoint family $(a_i)_{i\in I}$

$$\mu(\sup a_i) = \sum \mu(a_i) = \sup \Big\{ \sum_{i\in E} \mu(a_i),\ E \text{ finite subset of } I \Big\}.$$

Of course the first and the second condition are equivalent respectively to the conditions

$$\mu(\sup a_n) = \sup \mu(a_n) \text{ by } a_n \nearrow a$$

and

$$\mu(\sup a_i) = \sup \mu(a_i) \text{ by } a_i \nearrow a\,.$$

These things are well known.

Example 4. Let $X = [\![0, \Omega]\!]$ be the space of ordinals less than or equal to the first uncountable ordinal Ω. The point measure δ_Ω is a regular Borel measure on X. We see that this measure induces on the Boolean algebra $\mathfrak{R}_0(X)$ of all regular open sets a σ-additive measure, because every countable family of regular open neighborhoods of the limit point Ω has a non empty regular open infimum. But this measure on $\mathfrak{R}_0(X)$ is not completely additive, since the infimum of all regular open neighborhoods of Ω is zero in $\mathfrak{R}_0(X)$ and the measure of all these neighborhoods is identically 1.

Theorem 4. *Let μ be a positive normal Borel measure on an arbitrary topological space X. Then the function induced by μ on the Boolean algèbra of regular open sets is a completely additive measure. Two distinct normal Borel measures have distinct completely additive traces on the Boolean algebra of regular open sets. Every complete additive measure on the Boolean algebra of regular open sets is the trace of a normal Borel measure on X if the space X is a quasi-regular Baire space.*

Remark. This statement extends a theorem of Oxtoby [14] (cf. also Mibu [13]) to the case of normal measures while Oxtoby considers category measures. That every normal measure induces a σ-additive measure on $\mathfrak{R}_0(X)$ is also remarked in Hebert-Lacey [7].

Proof. Let $\mu \in \mathcal{N}^+(X)$; we restrict μ to its regular closed support. Then the trace of μ on $\mathfrak{R}_0(\operatorname{supp} \mu)$ is a σ-additive strictly positive measure. Therefore it is completely additive on $\mathfrak{R}_0(\operatorname{supp} \mu)$. But then the extension to $\mathfrak{R}_0(X)$ remains completely additive. For normal measures $\mu \in \mathcal{N}^+(X)$ we have $\mu(U) = \mu(\operatorname{Int} \overline{U})$, U open in X. By regularity of μ for a Borel set B:

$$\mu(B) = \inf \{\mu(H) \mid H \supset B,\, H \text{ regular open}\}\,.$$

Thus a normal measure is uniquely determined by its trace on $\mathfrak{R}_0(X)$.

The last part is a consequence of Theorem 2 in Oxtoby [14].

Theorem 5. *Let X, Y be regular Baire spaces and $f : X \to Y$ a proper irreducible map from X onto Y (this means that f is continuous, closed with compact fibres, and no closed subset $F \neq X$ is mapped onto Y). Then f gives in a natural way an isometric isomorphism from the space of normal measures on X onto the space of normal measures on Y.*

Remark. This statement generalizes the investigation of Hebert-Lacey on normal measures and the projective resolution.

Proof. It follows from Theorem 4 and our investigation about the behavior of regular open (closed) sets under proper irreducible maps [5], [6].

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