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In: Josef Novák (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Third Prague Topological Symposium, 1971. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1972. pp. 19--22.

Persistent URL: http://dml.cz/dmlcz/700711

UNIFORM CONTINUITY IN PARACOMPACT SPACES

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Let E_1 and E_2 be two nondiscrete paracompact Hausdorff spaces, $\mathfrak{U}_1$ and $\mathfrak{U}_2$ uniformities in E_1 and E_2, respectively, and $\mathfrak{U}_1 \otimes \mathfrak{U}_2$ the product uniformity in $E_1 \times E_2$. On $[0, 1]$ we consider the usual metric topology (thus there is a unique uniformity $\mathfrak{U}'$ in $[0, 1]$). We shall study the following question: Under what conditions every continuous function of $E_1 \times E_2$ (with the product topology) into $[0, 1]$ is a uniformly continuous map of $(E_1 \times E_2, \mathfrak{U}_1 \otimes \mathfrak{U}_2)$ into $([0, 1], \mathfrak{U}')$?

1. Preliminaries

Let E be a nondiscrete completely regular Hausdorff space.

Definition. The *index* of E is the least cardinal number for which there is a family (with this cardinality) of open subsets of E, whose intersection is not an open set. (Let us denote by m the index of E.)

Let E_1 and E_2 be two nondiscrete completely regular Hausdorff spaces. For $i = 1, 2$ let $\mathfrak{U}_i$ be a uniformity in E_i (i.e. compatible with the topology for E_i); $\mathfrak{U}_1 \otimes \mathfrak{U}_2$ denotes the product uniformity in $E_1 \times E_2$, i.e. the uniformity in $E_1 \times E_2$ for which the set $\{U_1 \otimes U_2 \mid U_1 \in \mathfrak{U}_1, U_2 \in \mathfrak{U}_2\}$, where $U_1 \otimes U_2 = \{((x_1, x_2), (y_1, y_2)) \mid (x_i, y_i) \in U_i, i = 1, 2\}$, is a basis.

Proposition. *Suppose $m > \aleph_0$ and every locally finite open covering of E has cardinality less than m. We have:*

1) *if E is a normal space, then every subset of E of cardinality m has an accumulation point (in E) and every open covering of E of cardinality m has a subcovering of cardinality less than m;*

2) *if E is a normal space, then every closed subset of E which is the intersection of at most m open subsets of E has a fundamental system $\mathfrak{B}$ of open neighborhoods, whose cardinality $|\mathfrak{B}|$ is less than or equal to m;*

3) *if m is the pseudoweight at some point $x \in E$, then m is the weight at the point x;*

4) *if E is a topological group and m is the pseudoweight at the neutral element of E, then E is paracompact.*

Proof. By the definition of index, m is a regular cardinal number. Since $m > \aleph_0$ and E is completely regular, every point of E has a fundamental system of neighborhoods, which are open-closed in E. (The closed G_δ-subsets of E are open.) The union of less than m closed subsets of E is closed in E.

Assertion 1) follows easily from the above consideration.

Assertions 2) and 3) are proved by using the same technique. We shall prove 3). Let us denote by μ the first ordinal number of cardinality m and by M the set of all ordinals less than μ. Since m is the pseudoweight at the point $x \in E$, there is a family $(V_i)_{i \in M}$ of neighborhoods of x, whose intersection is $\{x\}$. (We can and will suppose that the V_i are open-closed in E.) Put $W_0 = V_0$ and $W_i = \bigcap_{j<i} V_i$ for each $i \in M - \{0\}$. The family $(W_i)_{i \in M}$ is a fundamental system of neighborhoods of x. Indeed, let U be an open-closed neighborhood of x (x has a fundamental system of neighborhoods of this type). Consider the set $\{W_i - (W_{i'} \cup U) \mid i \in M - \{0\}\}$, where i' is the ordinal successor of i. It is a discrete collection of open-closed subsets of E. So there is $p \in M$ such that $W_i - (W_{i'} \cup U) = \emptyset$ for every $i \in M$, $i > p$. Thus $W_{p'} \subset U$. (If it were $t \in W_{p'}$ and $t \notin U$ there would be a minimal $k \in M$ such that $t \notin V_k$. By the construction of $W_{p'}$, k is greater than p; and t belongs to W_k and does not belong to $W_{k'}$; but W_k is contained in $W_{k'} \cup U$, which is a contradiction.)

Proof of 4). By virtue of assertion 3) the neutral element of E has a fundamental system $\mathfrak{B}$ of neighborhoods, with $|\mathfrak{B}| = m$. Since m is greater than $\aleph_0$ we can choose elements of $\mathfrak{B}$ such that $VV = V^{-1} = V$ for every $V \in \mathfrak{B}$. For each $V \in \mathfrak{B}$, $\{Vx \mid x \in E\}$ is a discrete open covering of E. The paracompactness of E follows from the fact that $\bigcup_{V \in \mathfrak{B}} \{Vx \mid x \in E\}$ is an open basis of the topology on E.

2. Main results

Let E_1 and E_2 be two nondiscrete completely regular Hausdorff spaces and m_1 and m_2 their indices.

Theorem 1. *Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be uniformities in E_1 and E_2, respectively. If every continuous function of $E_1 \times E_2$ (with the product topology) into $[0, 1]$ is a uniformly continuous map of $(E_1 \times E_2, \mathfrak{U}_1 \otimes \mathfrak{U}_2)$ into $([0, 1], \mathfrak{U}')$, then every locally finite open covering of E_i has cardinality less than m_j $(i, j = 1, 2)$.*

Proof. We shall prove, for instance, that every locally finite open covering of E_1 has cardinality less than m_2. It is sufficient to prove that every discrete family of nonempty open subsets of E_1 has cardinality less than m_2.

On the contrary, let us suppose that there exists a discrete family of nonempty open subsets of E_1, $(W_t)_{t \in T}$, whose cardinality $|T|$ is equal to m_2. There is a point $d \in E_2$ and a family of open neighborhoods of d, $(V_t)_{t \in T}$, such that $\bigcap_{t \in T} V_t$ is not a neighborhood of d. For each $t \in T$ we fix a point $a_t \in W_t$ and two continuous functions $f_t : E_1 \to [0, 1]$ and $g_t : E_2 \to [0, 1]$ satisfying the conditions:

1) $f_t(a_t) = 1$, $f_t(E_1 - W_t) = \{0\}$;
2) $g_t(d) = 1$, $g_t(E_2 - V_t) = \{0\}$.

The function g defined below is continuous (because the family $(W_t \times V_t)_{t \in T}$ is discrete in $E_1 \times E_2$):

$$g : E_1 \times E_2 \to [0, 1]$$
$$(x, y) \mapsto 0 \quad \text{if } (x, y) \text{ does not belong to } \bigcup_{t \in T} W_t \times V_t$$
$$(x, y) \mapsto f_t(x)\, g_t(y) \quad \text{if} \quad (x, y) \in W_t \times V_t, \quad t \in T.$$

By the hypothesis, there are $U_1 \in \mathfrak{U}_1$ and $U_2 \in \mathfrak{U}_2$, such that $(x, y) \in U_1$ and $(u, v) \in U_2$ imply $|g(x, u) - g(y, v)| < \frac{1}{4}$. But this is not possible, because then $U_2[d]$ would be contained in $\bigcap_{t \in T} V_t$. ($(a_t, a_t) \in U_1$ and $(u, d) \in U_2$ imply $g(a_t, u) \geqq \frac{3}{4}$, so $u \in V_t$.) The proof is completed; notice that $m_1 = m_2$.

Remark 1. Suppose $m_1 = m_2 = p$. If E_1 and E_2 are paracompact and every locally finite open covering of E_i $(i = 1, 2)$ has cardinality less than p, then $E_1 \times E_2$ is paracompact and, further, every locally finite open covering of $E_1 \times E_2$ has cardinality less than p. So if $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are the universal uniformities in E_1 and E_2, $\mathfrak{U}_1 \otimes \mathfrak{U}_2$ is the universal uniformity in $E_1 \times E_2$.

The next theorem follows easily.

Theorem 2. *Let E be a nondiscrete paracompact space and m the index of E. The following conditions are equivalent:*

1) *there is a uniformity $\mathfrak{U}$ in E such that every continuous function of $E \times E$ (with the product topology) into $[0, 1]$ is a uniformly continuous map of $(E \times E, \mathfrak{U} \otimes \mathfrak{U})$ into $([0, 1], \mathfrak{U}')$;*

2) *every locally finite open covering of E has cardinality less than m;*

3) *there is a uniformity $\mathfrak{U}$ in E such that $\mathfrak{U} \otimes \ldots \otimes \mathfrak{U}$ (n times) is the universal uniformity in the product topological space E^n, $n = 2, 3, \ldots$;*

4) *there is a uniformity $\mathfrak{U}$ in E such that $\mathfrak{U} \otimes \mathfrak{U}$ is the universal uniformity in the product topological space $E \times E$.*

Hint. If E satisfies the condition 2), then E^n is a paracompact space for each $n = 2, 3, \ldots$ On the other hand, it is well-known that if X is a paracompact space, the set $\{\bigcup_{Y \in \alpha} Y \times Y \mid \alpha \text{ is a locally finite open covering of } X\}$ is a basis of the universal uniformity in X.

Remark. The implication 2) $\Rightarrow$ 4) is a particular case of Theorem 35 ([7], p. 137).

For topological groups we have the following theorem ([2]):

Theorem 3. *Suppose E is a paracompact topological group. E satisfies the condition* 2 *of Theorem* 2 *if and only if the right uniformity in E is the universal uniformity in E.*

For other results in the same area see, for instance, [3], [5], [7] and [8]. Professor L. Nachbin also investigated a similar question for metric spaces.

References

[1] *O. T. Alas:* Paracompact topological groups and uniform continuity. (To appear.)
[2] *O. T. Alas:* Normal topological groups and universal uniformities. An. Acad. Brasil. Ci. *42* (1970), 411—413.
[3] *M. Atsuji:* Uniform continuity of continuous functions on uniform spaces. Canad. J. Math. *13* (1961), 654—663.
[4] *N. Bourbaki:* Éléments de Mathématique. Topologie Générale. Livre 3, chapitres 1 et 2. Hermann, Paris, 1965.
[5] *W. W. Comfort and K. A. Ross:* Pseudocompactness and uniform continuity in topological groups. Pacific J. Math. *16* (1966), 483—496.
[6] *R. Engelking:* Outline of General Topology. North-Holland Publishing Co, Amsterdam, 1968.
[7] *J. R. Isbell:* Uniform spaces. Math. Surveys *12*. Amer. Math. Soc., 1964.
[8] *N. Onuchic:* On the Nachbin uniform structure. Proc. Amer. Math. Soc. *11* (1960), 177—179.