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SEPARATION AND CONNECTEDNESS

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Our purpose is to study families of sets having properties similar to those of the family of all connected sets of a topological space, as well as the relation of such families to binary relations called separations. Let us begin with the definition of the latter concept.

<u>Definition I.</u> The relation σ between the subsets of a set X is said to be a <u>separation</u> if

$$\emptyset \, \sigma \, A, \quad A \, \sigma \, \emptyset \quad \text{for} \quad A \subset X,$$

$$A \, \sigma \, B \quad \text{implies} \quad A \cap B = \emptyset,$$

$$A \, \sigma \, B, \quad A' \subset A, \quad B' \subset B \quad \text{implies} \quad A' \, \sigma \, B'.$$

The separation σ is said to be <u>symmetrical</u> if $A \, \sigma \, B$ implies $B \, \sigma \, A$.

The notion of separation is nothing else than another formulation of the notion of semi-topogenous order. In fact, as it is well-known ([1], p. 7):

<u>Definition II.</u> The relation $<$ between the subsets of a set X is said to be a semi-topogenous order if

$$\emptyset < \emptyset, \quad X < X,$$

$$M < N \quad \text{implies} \quad M \subset N,$$

$$M \subset M' < N' \subset N \quad \text{implies} \quad M < N.$$

The semi-topogenous order $<$ is said to be symmetrical if

$$M < N \quad \text{implies} \quad X - N < X - M.$$

The above mentioned equivalence of separation and semi-topogenous order is contained in the following statements:

<u>Theorem I.</u> If $<$ is a semi-topogenous order and the relation σ is defined by

(1) $$A \, \sigma \, B \quad \text{iff} \quad A < X - B,$$

then σ is a separation. Conversely if σ is a separation and the relation $<$ is defined by

(2) $$M < N \quad \text{iff} \quad M \, \sigma \, X - N,$$

then $<$ is a semi-topogenous order.

<u>Proof.</u> σ defined by (1) is a separation:

$$\emptyset \subset \emptyset < \emptyset \subset X - A \Rightarrow \emptyset\ \sigma\ A;$$

$$A \subset X < X \subset X - \emptyset \Rightarrow A\ \sigma\ \emptyset;$$

$$A\ \sigma\ B \Leftrightarrow A < X - B \Rightarrow A \subset X - B \Rightarrow A \cap B = \emptyset;$$

$$A\ \sigma\ B,\quad A' \subset A,\quad B' \subset B \Rightarrow A' \subset A < X - B \subset X - B' \Rightarrow A' < X - B' \Leftrightarrow$$

$$\Leftrightarrow A'\ \sigma\ B'.$$

On the other hand $<$ defined by (2) is a semi-topogenous order:

$$\emptyset\ \sigma\ X - \emptyset \Rightarrow \emptyset < \emptyset;\quad X\ \sigma\ X - X \Rightarrow X < X;$$

$$M < N \Leftrightarrow M\ \sigma\ X - N \Rightarrow M \cap (X - N) = \emptyset \Leftrightarrow M \subset N;$$

$$M \subset M' < N' \subset N \Leftrightarrow M \subset M',\quad M'\ \sigma\ X - N',\quad X - N' \supset X - N \Rightarrow M\ \sigma\ X - N \Leftrightarrow$$

$$\Leftrightarrow M < N.$$

Theorem II. If σ is obtained by (1) from $<$, and $<$ is obtained by (2) from this σ, then $<' = <$; conversely if $<$ is obtained by (2) from σ and σ' from this $<$ by (1), then $\sigma' = \sigma$.

The relations $<$ and σ satisfying (1) and (2) are said to be associated with each other.

E. g. if, in a topological space X, the semi-topogenous order $<$ is defined by

$$M < N \text{ iff } M \subset \operatorname{int} N,$$

then the separation σ_0 associated with $<$ is given by

(3) $\quad A\ \sigma_0\ B$ iff $A \subset \operatorname{int}(X - B) = X - \bar{B}$ iff $A \cap \bar{B} = \emptyset$.

On the other hand, let $<$ be defined by

$$M < N \text{ iff } M \subset \operatorname{int} N \text{ and } \bar{M} \subset N,$$

then σ associated with $<$ will be given by

(4) $\quad A\ \sigma\ B$ iff $A \cap \bar{B} = \bar{A} \cap B = \emptyset$.

Concerning symmetrical separations we can say:

Theorem III. The separation σ is symmetrical iff the semi--topogenous order $<$ associated with it is symmetrical.

Proof. If σ is symmetrical, then

$$M < N \Leftrightarrow M\ \sigma\ X - N \Rightarrow X - N\ \sigma\ M \Leftrightarrow X - N < X - M;$$

if $<$ is symmetrical, then

$$A\ \sigma\ B \Leftrightarrow A < X - B \Rightarrow B < X - A \Leftrightarrow B\ \sigma\ A.$$

Let us now consider families of sets defined by some conditions obviously fulfilled for the system of all connected sets in a topological space:

Definition III. The system $\mathcal{L}$ of subsets of a set X is said to be a connectivity if

$$\emptyset \in \mathcal{L}, \quad \{p\} \in \mathcal{L} \quad \text{for } p \in X,$$
$$C_i \in \mathcal{L} \quad (i \in I), \quad \bigcap_{i \in I} C_i \neq \emptyset \text{ implies } \bigcup_{i \in I} C_i \in \mathcal{L}.$$

Connectivities can be obtained from separations in the following way:

__Theorem IV.__ Let σ be a separation on X, and let $C \in \mathcal{L}$ hold iff

$$C = A \cup B, \quad A \,\sigma\, B \text{ implies } A = \emptyset \text{ or } B = \emptyset;$$

then $\mathcal{L}$ is a connectivity called the connectivity __induced__ by σ.

__Proof.__ $\emptyset = A \cup B$ implies $A = B = \emptyset$, hence $\emptyset \in \mathcal{L}$.

$\{p\} = A \cup B$, $A \,\sigma\, B$ implies, by $A \cap B = \emptyset$, $A = \{p\}$, $B = \emptyset$ or $B = \{p\}$, $A = \emptyset$, hence $\{p\} \in \mathcal{L}$.

$\bigcup_{i \in I} C_i = C = A \cup B$, $A \,\sigma\, B$, $p \in \bigcap_{i \in I} C_i$ implies e. g. $p \in A$; then, for every i, $C_i = (A \cap C_i) \cup (B \cap C_i) = A_i \cup B_i$ where evidently $A_i \,\sigma\, B_i$, $p \in A_i \Rightarrow A_i \neq \emptyset$, thus $C_i \in \mathcal{L}$ implies $B_i = \emptyset$ and $B = \bigcup_{i \in I} B_i = \emptyset$.

E. g. if, in a topological space, σ is defined by (4), then this separation induces the family of all connected sets in the usual sense. This σ is evidently symmetrical. Its relation to the separation σ_0 defined by (3) can be given as follows:

(5) $$A \,\sigma\, B \text{ iff } A \,\sigma_0\, B \text{ and } B \,\sigma_0\, A.$$

In general, it is clear that if σ_0 is a separation, then the relation σ defined by (5) is the finest symmetrical separation coarser than σ_0. Here the separation σ_1 is said to be __coarser__ than the separation σ_2 (or σ_2 finer than σ_1) if $A \,\sigma_1\, B$ implies $A \,\sigma_2\, B$.

Clearly the relation of being finer (coarser) is a partial ordering among the separations.

Conversely to Theorem IV, if a connectivity $\mathcal{L}$ is given, we can always construct a separation σ which induces $\mathcal{L}$. More precisely:

__Theorem V.__ Let $\mathcal{L}$ be a connectivity and, by definition, let $A \,\sigma'\, B$ hold iff there exists no $C \in \mathcal{L}$ such that

$$C \subset A \cup B, \quad A \cap C \neq \emptyset \neq B \cap C.$$

The relation σ' defined here is a symmetrical separation which induces $\mathcal{L}$.

<u>Proof.</u> σ' is evidently symmetrical.

σ' is a separation. $\emptyset\ \sigma'\ B$ and $A\ \sigma'\ \emptyset$ by $\emptyset \cap C = \emptyset$; $A\ \sigma'\ B$ implies $A \cap B = \emptyset$, otherwise there would be a set $C = \{p\} \subset A \cup B$ with the property $C \in \mathcal{L}$, $C \cap A \neq \emptyset \neq C \cap B$; if $A\ \sigma'\ B$, $A' \subset A$, $B' \subset B$ but $A'\ \sigma'\ B'$ would not hold, then there would be a set $C' \subset A' \cup B'$ such that $C' \in \mathcal{L}$, $C' \cap A' \neq \emptyset \neq C' \cap B'$ and then $C' \subset A \cup B$, $C' \cap A \supset C' \cap A' \neq \emptyset$, $C' \cap B \supset C' \cap B' \neq \emptyset$ in contradiction to $A\ \sigma'\ B$.

σ' induces $\mathcal{L}$. To prove this let $\mathcal{L}'$ denote the connectivity induced by σ' according to Theorem IV. We prove that $\mathcal{L} = \mathcal{L}'$. $\mathcal{L} \subset \mathcal{L}'$ since $C = A \cup B$, $A\ \sigma'\ B$, $A \neq \emptyset$, $B \neq \emptyset$ would be in contradiction with $C \in \mathcal{L}$. On the other hand $\mathcal{L}' \subset \mathcal{L}$. Let $C' \in \mathcal{L}'$. $\mathcal{L}$ is a connectivity, therefore if $C' = \emptyset$ then $C' \in \mathcal{L}$. If $C' \neq \emptyset$, choose $p \in C'$. Set

(*) $$A = \bigcup \{C_i : p \in C_i \subset C',\ C_i \in \mathcal{L}\}.$$

Then $A \in \mathcal{L}$, since $\mathcal{L}$ is a connectivity, and $p \in A \subset C'$ by $\{p\} \in \mathcal{L}$. Let $B = C' - A$. Prove first that $A\ \sigma'\ B$. In fact, for a set $C \in \mathcal{L}$ with $C \subset A \cup B = C'$, $C \cap A \neq \emptyset$, we have $p \in A \cup C \in \mathcal{L}$, $A \cup C \subset C'$ hence by (*) $A \cup C \subset A$, $C \subset A$, $C \cap B = \emptyset$. Consequently $A\ \sigma'\ B$ and $A \neq \emptyset$ implies $B = \emptyset$, $C' = A \in \mathcal{L}$, and therefore $\mathcal{L}' \subset \mathcal{L}$.

In general, a connectivity can be induced by several separations. However, that one defined in Theorem V is the finest among them:

<u>Theorem VI.</u> If $\mathcal{L}$ is a connectivity and σ' is the separation defined in Theorem V, then σ' is the finest one among all separations inducing $\mathcal{L}$.

<u>Proof.</u> If σ induces $\mathcal{L}$ and $A\ \sigma\ B$, $C \subset A \cup B$, $C \cap A \neq \emptyset \neq C \cap B$, then $C \cap A\ \sigma\ C \cap B$ and $(C \cap A) \cup (C \cap B) = C$ imply $C \notin \mathcal{L}$, so that $A\ \sigma'\ B$.

The fact that two separations distinct from each other can induce the same connectivity may be illustrated by the following example. Let X be the real line with the usual topology, and let σ denote the separation defined by (4). Then the connectivity $\mathcal{L}$ induced by σ consists of all intervals. With this $\mathcal{L}$, consider the separation σ' defined in Theorem V. Both σ and σ' induce $\mathcal{L}$, however $\sigma \neq \sigma'$. In fact, if

$$A = [0,\ 1] \cap Q, \quad B = (1,\ 2) \cap Q$$

(Q is the set of rational numbers), then $A\ \sigma\ B$ does not hold since $1 \in A \cap \bar{B}$, but clearly $A\ \sigma'\ B$.

The answer to the following open problem would be of some interest. Characterize those connectivities $\mathcal{L}$ whose elements coincide with all connected sets of a topological space X, or the same question for some special class of topological spaces.

REFERENCES

[1] Á. Császár: Foundation of General Topology. Pergamon Press, Oxford-London-New York-Paris, 1963.