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GENERATION OF BAIRE SETS

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Let X be a uniform space and U(X) the Banach space of all bounded real-valued uniformly continuous functions on X with the supremum norm. Let

$$Z_0(X) = \{Z(f) : f \in U(X)\},$$

where $Z(f) = \{x \in X : f(x) = 0\}$. Then (corollary 1) each set of the form

$$X \setminus Z(f), \quad f \in U(X),$$

is the countable disjoint union of countable intersections of countable disjoint unions of sets in $Z_0(X)$. This result is best possible, since no non-empty proper open subset of a connected compact Hausdorff space X is the countable disjoint union of closed subsets of X; consequently for X = [0,1], the open unit interval is not the countable intersection of countable disjoint unions of closed subsets of X [6,p.173].

It is then shown (corollary 2) that the smallest family of sets containing $Z_0(X)$ and closed under countable disjoint unions and countable intersections is also closed under countable unions and complementation. This result is used to show (theorem 6) that the smallest family of subsets of a completely Hausdorff analytic space X (definitions below) containing the closed sets and closed under countable intersections coincides with the family of sets representable by means of the disjoint Souslin operation applies to the closed sets. The analogous result (theorem 7) is obtained for the family of zero sets of continuous real-valued functions on X. There is a (non-analytic) separable metric space where this result fails.

If the topological space X is a completely Hausdorff descriptive Borel space (in particular, compact), then (theorem 9) the above two families obtained from the closed sets also coincide with the family of descriptive Borel subsets. Several related results are given.

§ 1. NOTATION

Let X be a set and $\underset{\sim}{H}$ a family of subsets of X.

$\underset{\sim}{H}_\delta$ = countable intersections of sets in $\underset{\sim}{H}$.

$\underset{\sim}{H}_\sigma$ = countable unions of sets in $\underset{\sim}{H}$.

$\underset{\sim}{H}_{\sigma_d}$ = countable disjoint unions of sets in $\underset{\sim}{H}$.

$\underset{\sim}{H}_c$ = complements (relative to X) of sets in $\underset{\sim}{H}$.

$B(\underset{\sim}{H})$ = smallest family $\underset{\sim}{H}^* \supseteq \underset{\sim}{H}$ such that $\underset{\sim}{H}^* = \underset{\sim}{H}^*_\sigma = \underset{\sim}{H}^*_\delta$.

$B_d(\underset{\sim}{H})$ = smallest family $\underset{\sim}{H}^* \supseteq \underset{\sim}{H}$ such that $\underset{\sim}{H}^* = \underset{\sim}{H}^*_{\sigma_d} = \underset{\sim}{H}^*_\delta$.

$B_c(\underset{\sim}{H})$ = smallest family $\underset{\sim}{H}^* \supseteq \underset{\sim}{H}$ such that $\underset{\sim}{H}^* = \underset{\sim}{H}^*_{\sigma_d} = \underset{\sim}{H}^*_c$.

$S(\underset{\sim}{H})$ = sets representable in the form

$$\bigcup_\sigma \bigcap_{n=1}^{\infty} H_{\sigma|n}\ , \quad H_{\sigma|n} \in \underset{\sim}{H}$$

where the union is over all sequences $\sigma = (\sigma_1, \sigma_2, \ldots)$ of positive integers and $\sigma|n$ denotes $\sigma_1, \sigma_2, \ldots, \sigma_n$.

$S_d(\underset{\sim}{H})$ = sets in $S(\underset{\sim}{H})$ which have representations as above such that for all $\sigma \neq \tau$,

$$\bigcap_n H_{\sigma|n} \cap \bigcap_n H_{\tau|n} = \emptyset\ .$$

The following diagram of inclusions holds

$$\begin{array}{ccccc} & & \underset{\sim}{H}_{\sigma_d} & \subseteq & \underset{\sim}{H}_\sigma \\ & & \cap\!| & & \cap\!| \\ \underset{\sim}{H}_\delta & \subsetneq & B_d(\underset{\sim}{H}) & \subseteq & B(\underset{\sim}{H}) \\ & & \cap\!| & & \cap\!| \\ & & S_d(\underset{\sim}{H}) & \subsetneq & S(\underset{\sim}{H}) \end{array}$$

If X is a uniform space, then U(X) denotes the space of all bounded uniformly continuous real-valued functions on X. The Baire sets of multiplicative class 0, denoted $Z_o(X)$, are the zero sets of functions in U(X). The sets of additive class 0, denotes $CZ_o(X)$, are the complements of the sets in $Z_o(X)$. Define inductively for each countable ordinal α the sets of multiplicative class α, denoted $Z_\alpha(X)$, to be the countable intersections of sets in $\bigcup_{\xi<\alpha} CZ_\alpha(X)$, and define the sets of additive class α, denoted $CZ_\alpha(X)$, to be the complements of the sets in $Z_\alpha(X)$. For each α, $0 \leqslant \alpha < \Omega$ (where Ω denotes the first uncountable ordinal), we have

$$CZ_\alpha(X)_\delta = Z_{\alpha+1}(X) \quad , \quad Z_\alpha(X)_\sigma = CZ_{\alpha+1}(X) \quad ,$$

$$CZ_\alpha(X) \subseteq CZ_{\alpha+1}(X) \quad , \quad Z_\alpha(X) \subseteq Z_{\alpha+1}(X) \quad ,$$

$$CZ_\Omega(X) = Z_\Omega(X) = B(Z_\Omega(X)) = Z_\Omega(X)_c \ .$$

This definition of additive and multiplicative classes differs slightly from that used in [4], since there is no concern here for the relation between the Baire sets and Baire functions of class α. The above definition is simpler for the purposes of this paper.

§ 2. PRELIMINARY RESULTS

A. Following are two abstract formulations of a classical theorem of N.Lusin [5,p. 348, Th.3]:

1. (C.A.Rogers[9]). If $\underset{\sim}{H}$ is a family of subsets of a set X, if $\underset{\sim}{H}' \supseteq \underset{\sim}{H}$ is the smallest family satisfying a) $H_2 \setminus H_1 \in \underset{\sim}{H}'$ whenever $H_1, H_2 \in \underset{\sim}{H}'$, and b) $\underset{\sim}{H}' = \underset{\sim}{H}'_\sigma$, and if $\underset{\sim}{H}'' \supseteq \underset{\sim}{H}$ is the smallest family satisfying a) $H_2 \setminus H_1 \in \underset{\sim}{H}''$ whenever $H_1, H_2 \in \underset{\sim}{H}$, and b) $\underset{\sim}{H}'' = B_d(\underset{\sim}{H}'')$, then $\underset{\sim}{H}' = \underset{\sim}{H}''$.

2. (R.O. Davis). If $\underset{\sim}{H}'$ is a family of subsets of a set X such that $\underset{\sim}{H}' = B_d(\underset{\sim}{H}')$, then $\underset{\sim}{H}' \cap \underset{\sim}{H}'_c = B_d(\underset{\sim}{H}' \cap \underset{\sim}{H}'_c)$. Consequently, (Z.Frolík [2,p.407]) if $\underset{\sim}{H}$ is a family of subsets of X such that $\underset{\sim}{H}_c \subseteq B_d(\underset{\sim}{H})$, then $B_d(\underset{\sim}{H}) = B_d(\underset{\sim}{H})_c = B(\underset{\sim}{H})$.

B. The two propositions (and their proofs) in A obscure the step-by-step phenomenon in the transfinite generation processes. The following formulation clarifies this aspect: (Lusin[5,p.348,Th.2]) Let X be a uniform space.

a) If $\alpha > 0$ is a countable ordinal, then $Z_\alpha(X)_{\sigma_d} = Z_\alpha(X)_\sigma = CZ_{\alpha+1}(X)$.

b) If each set in $CZ_o(X)$ is the countable union of sets in $Z_o(X) \cap CZ_o(X)$, then each set in $CZ_o(X)$ is the countable disjoint union of sets in $Z_o(X) \cap CZ_o(X)$ and $Z_o(X)_{\sigma_d} = Z_o(X)_\sigma = CZ_1(X)$. This result says that, in the generation of Baire sets by the transfinite iteration of the operations of countable unions and countable intersections, the countable unions can be replaced by countable disjoint unions in every step, except the passage from $Z_o(X)$ to $Z_1(X)$. Even in this step countable disjoint unions suffice if $Z_o(X)$ are the closed sets of a separable 0-dim metric space X. This does not characterize 0-dim spaces among separable metric spaces, as there is a 1-dim space in which countable disjoint unions suffice [5, p. 299]. Countable disjoint unions do not suffice if $Z_o(X)$ are the zero sets of continuous real-valued functions on a connected compact Hausdorff space (with more than one point) X. In fact,

Theorem 1. A compact Hausdorff space X is 0-dimensional if and only if $Z_o(X)_{\sigma_d} = Z_o(X)_\sigma$.

C. From A and B above it is seen that for a uniform space X

$$B_d(CZ_o(X)) = CZ_\Omega(X) = Z_\Omega(X)$$

and

$$B(Z_o(X)) = Z_\Omega(X) = CZ_\Omega(X).$$

We will see in the next section that

$$B_d(Z_o(X)) = Z_\Omega(X).$$

We also have

$$B_c(Z_o(X)) = B_c(CZ_o(X)) = Z_\Omega(X).$$

This follows from the more general result (T.Neubrunn[7]) :

If $\underset{\sim}{H}$ is a family of subsets of a set X such that either

1) $H_1 \cap H_2 \in B_c(\underset{\sim}{H})$ whenever $H_1, H_2 \in \underset{\sim}{H}$, or

2) $H_1 \setminus H_2 \in B_c(\underset{\sim}{H})$ whenever $H_1, H_2 \in \underset{\sim}{H}$,

then $B(B_c(\underset{\sim}{H})) = B_c(\underset{\sim}{H})$.

§ 3. BAIRE SETS

Consider the real line R with the usual uniformity (any uniformity compatible with the topology will do). We have $Z_o(R)$ equal to the family of all closed subsets of R.

Theorem 2. If $a,b \in R$, $a < b$, then the open interval (a,b) is contained in $Z_o(R)_{\sigma_d\delta\sigma_d}$.

Remark: The open interval (a,b) does not belong to $Z_o(R)_{\sigma_d\delta}$ [5, p. 173].

Corollary 1. For a uniform space X

$$CZ_o(X) \subseteq Z_o(X)_{\sigma_d\delta\sigma_d}.$$

Corollary 2. For a uniform space X we have $B_d(Z_o(X)) = Z_\Omega(X)$. Consequently, $B_d(Z_o(X))$ is closed under complementation and countable unions.

Corollary 3. If X is a metric space and F is the family of all closed subsets of X, then $B_d(F)$ is the family of all Borel subsets of X.

Recall that the Borel sets of a topological space X are the members of the smallest family of sets containing the closed sets and closed under complementation and countable unions (and, consequently, countable intersections); and that the Baire sets of X are the members of the smallest family of sets containig the zero sets of continuous real-valued function and closed under countable unions and countable intersections (and, consequently, complementation).

Corollary 4. If X is a topological space and $\underset{\sim}{Z}$ is the family of zero sets of continuous real-valued functions on X, then $B_d(\underset{\sim}{Z})$ is the family of Baire sets in X.

It is known [3] that a family $\underset{\sim}{H}$ of subsets of a set X is equal to $Z_o(X)$ for some uniformity on X if and only if the following conditions are satisfied:

1) $\phi, X \in \underset{\sim}{H}$,

2) $H_1 \cup H_2 \in \underset{\sim}{H}$ whenever $H_1, H_2 \in \underset{\sim}{H}$,

3) $\bigcap_{n=1}^{\infty} H_n \in \underset{\sim}{H}$ whenever $(H_n) \subseteq \underset{\sim}{H}$,

4) If $H_1, H_2 \in \underset{\sim}{H}$ and $H_1 \cap H_2 = \phi$, then there are $H_3, H_4 \in \underset{\sim}{H}$ such that $(X \setminus H_3) \cap (X \setminus H_4) = \phi$ and

$$H_1 \subseteq X \setminus H_3 \quad , \quad H_2 \subseteq X \setminus H_4 \ ,$$

5) $\underset{\sim}{H}_c \subseteq \underset{\sim}{H}_\sigma$.

If we omit condition, 5) then theorem 2 no longer holds for $\underset{\sim}{H}$, since the closed sets in a compact space satisfy 1) to 4) and, in general, there are closed sets whose complements are not Lindelöf, and so can not belong to $S(\underset{\sim}{H}) \supseteq \underset{\sim}{H}_{\sigma_d \delta \sigma_d}$. Likewise theorem 3 does not hold in the absence of condition 5), since there is a compact space X containing a σ-compact subset Y such that

$$Y \in B(\underset{\sim}{F}(X)) \setminus B_d(\underset{\sim}{F}(X)) \quad ,$$

where $\underset{\sim}{F}(X)$ denotes the family of closed subsets of X (see[2, p.427].)

If we omit conditions 1) and 3) we obtain:

Theorem 3. If $\underset{\sim}{H}$ is a family of subsets of a set X which satisfies conditions 2), 4) and 5), then $\underset{\sim}{H}_c \subseteq \underset{\sim}{H}_{\delta\sigma_d\delta\sigma_d}$. Consequently, $B_d(\underset{\sim}{H})$ is closed under complementation and countable unions.

If we omit conditions 1) and 4) and either 2) or 3) we can prove a weaker, though analogous, result.

Theorem 4. Let $\underset{\sim}{H}$ be a family of subsets of a set X such that $\underset{\sim}{H}_c \subseteq \underset{\sim}{H}_\sigma$.

If either

1) $H_1 \cup H_2 \in \underset{\sim}{H}$ whenever $H_1, H_2 \in \underset{\sim}{H}$, or

2) $H_1 \cap H_2 \in \underset{\sim}{H}$ whenever $H_1, H_2 \in \underset{\sim}{H}$,

then $B(\underset{\sim}{H})_c = B(\underset{\sim}{H}) \subseteq S_d(\underset{\sim}{H})$.

Remark. In general $B(\underset{\sim}{H}) \neq S_d(\underset{\sim}{H})$ in Theorem 4, as there is a separable metric space such that

$$B(\underset{\sim}{F}(X)) \notin S_d(\underset{\sim}{F}(X)) \cap S_d(\underset{\sim}{F}(X))_c.$$

§ 4. ANALYTIC AND DESCRIPTIVE BOREL SETS.

Let $N = \{1,2,\ldots\}$ and N^N be the product of N copies of the discrete space N with the product topology. Let X and Y be topological spaces, and $\underset{\sim}{K}(X)$, and $\underset{\sim}{F}(X)$, and $Z_o(X)$ denote the families of compact, closed and zero sets of continuous real-valued functions on X, respectively. The families $Z_\alpha(X)$ and $CZ_\alpha(X)$ are then defined as before.

A map

$$F : X \to \underset{\sim}{K}(Y)$$

is upper semi-continuous if

$$F^{-1}(U) \equiv \{x \in X : F(x) \subseteq U\}$$

is open in X for every open set U in Y.

A Hausdorff space X is called analytic if there is an upper semi-continuous map

$$F : N^N \to \underset{\sim}{K}(X)$$

such that $X = \cup\{F(\sigma) : \sigma \in N^N\}$, and is called descriptive Borel [8] (or Borelian [1]) if, in addition, $\sigma \neq \tau$ ($\sigma, \tau \in N^N$) implies that

$$F(\sigma) \cap F(\tau) = \emptyset .$$

A topological space is called completely Hausdorff if the continuous real-valued functions separate the points.

<u>Theorem 5.</u> If X is a completely Hausdorff analytic space, then

$$S_d(\underset{\sim}{F}(X)) \subseteq \{F \cap C : F \in \underset{\sim}{F}(X),\ C \in Z_\Omega(X)\}_{\sigma_d \delta}.$$

<u>Theorem 6.</u> If X is a completely Hausdorff analytic space, then

$$B_d(\underset{\sim}{F}(X)) = S_d(\underset{\sim}{F}(X)) .$$

<u>Theorem 7.</u> If X is a analytic space, then the family of Baire sets equals

$$B_d(Z_o(X)) = S_d(Z_o(X)) .$$

<u>Theorem 8.</u> If X is a completely Hausdorff space and $B \subseteq X$ is descriptive Borel,then

$$B \in B_d(\underset{\sim}{F}(X)) .$$

<u>Theorem 9.</u> If X is a completely Hausdorff descriptive Borel space,then

$$\{B \subseteq X : B \text{ is descriptive Borel}\} = B_d(\underset{\sim}{F}(X)).$$

<u>Theorem 10.</u>The following conditions are equivalent for a completely regular space X:

1) X is descriptive Borel,

2) $X \in B_d(\underset{\sim}{F}(Y))$ for any completely Hausdorff space Y containing X,

3) $X \in B_d(\underset{\sim}{F}(\beta X))$, where βX denotes the Stone-Čech compactification of X.

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