Toposym 4-B

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Problems and remarks on contractibility of curves

PROBLEMS AND REMARKS ON CONTRACTIBILITY OF CURVES

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Although there are many papers and books which concern the notion of contractibility, and although the theory of contractible spaces is well known (see e.g. [12] and [15]), general methods are not very useful for investigation of contractibility of some special spaces, e.g. of curves. There are only a few papers concerning contractibility of curves; no internal characterization of the class of contractible curves is known at present and merely several conditions either necessary or sufficient have appeared in the literature. Thus it is to be desired that some recent result should be put together with open problems and some unanswered conjectures.

All spaces considered are assumed to be metric. The term continuum means a compact connected space. A property of a continuum X is called to be hereditary if each subcontinuum of X has this property. A continuum X is said to be arcwise connected if every two points a, b of X can be joined by an arc ab in X. A continuum X is called unicoherent if for each two subcontinua A and B of X such that $X = A \cup B$ the intersection $A \cap B$ is connected. A dendroid means an arcwise connected and hereditarily unicoherent continuum. A point p of an arcwise connected space X is called a ramification point of X provided there are three arcs pa, pb, pc such that p is the only common point of them. A dendroid which has only one ramification point is called a fan. One-dimensional continuum is called a curve. It is well known ([4], (48), p. 239) that every dendroid is a curve. A mapping means a continuous transformation. Let I denote the closed unit interval [0,1] of reals. A mapping $H: X \times I \longrightarrow Y$ is called a homotopy. If $X \subset Y$ and if $H(x,0) = x$ for each $x \in X$, then the homotopy H is said to be deformation of X in Y. Furthermore, if for each $x \in X$ the point $H(x,1)$ is the same (i.e., if $H(.,1)$ is a constant mapping), then the deformation H is called a contraction of X in Y. They are simply called deformations and contractions of X if $Y = X$. If a contraction of X (in Y) exists, then X is said to be contractible (in Y) (see [12], I, 8, p. 11 and 12; cf. [16], §54, I, p. 360; IV, V and VI, p. 368-375). Two mappings f, g: $X \longrightarrow Y$ are called to be homotopic if there exists a homotopy $H: X \times I \longrightarrow Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for each $x \in X$. We say that a curve is acyclic provided each mapping of it into the circle is homotopic to a constant mapping. It is known ([17], (2.3), p. 52) that if a curve is acyclic , then it is hereditarily unicoherent (the inverse implication is not true ([6], p. 218) however it

holds for hereditarily decomposable continua ([6], XI, p. 217).

Proposition 1. Every contractible curve is a dendroid.

Indeed, let X be a contractible curve. Hence X is arcwise connected ([16], §54, VI, Theorem 1, p. 374). Furthermore, X is contractible with respect to the circle, i.e., every mapping of X into the circle is homotopic to a constant ([16], §57, I, Theorem 9 (i), p. 435). This means that X is acyclic, and therefore ([17], (2.3), p. 52) it is hereditarily unicoherent. Thus X is a dendroid.

A non-empty subset A of a space X is said to be homotopically fixed (see [10], Definition 2) if for every deformation H: $X \times I \longrightarrow X$ we have $H(A \times I) \subset A$. The following two propositions are proved in [10].

Proposition 2. If a space X contains a non-empty subset A and a proper subset B such that for every deformation H: $X \times I \longrightarrow X$ we have $H(A \times I) \subset B$, then X is not contractible.

Proposition 3. If a space X contains a proper subset A which is homotopically fixed, then X is not contractible.

It is well known that a contractible space must be arcwise connected ([16], §54, VI, Theorem 1, p. 374). For curves however a stronger result can be shown. Recall that a subset X of a metric space is said to be uniformly arcwise connected (see [5], p. 193; [7], p. 12 and [8], p. 316) if it is arcwise connected and if for every number $\varepsilon > 0$ there is a positive integer k such that every arc A in X contains points a_0, $a_1, \ldots, a_k$ with the properties $A = \bigcup \{a_i a_{i+1} : i = 0, 1, \ldots, k-1\}$ and diam $(a_i a_{i+1}) < \varepsilon$ for every $i = 0, 1, \ldots, k-1$. The following is an immediate consequence of Proposition 1 and of Theorem 3 of [9], p. 94.

Proposition 4. Every contractible curve is uniformly arcwise connected.

It is known that a dendroid is uniformly arcwise connected if and only if it is a continuous image of the cone over the Cantor set (this is a consequence of a stronger result, see [14], Corollary 3.6, p.322). Using this, Proposition 4 can be reformulated as

Proposition 5. Every contractible curve is a continuous image of the cone over the Cantor set.

The converse obviously does not hold.

Recall that a dendroid X is said to be smooth ([8], p. 298) if it contains a point p such that given any sequence of points a_n in X with $\lim_n a_n = a$, it follows that the sequence of arcs $p a_n$ is convergent, and Lim $p a_n = pa$. For any fixed number $t \in I$, let i_t be a mapping of a space X onto $X \times \{t\}$ defined by $i_t(x) = (x, t)$ for every $x \in X$; recall that a mapping H: $X \times I \longrightarrow X$ is said to be a retracting homotopy if the composite $H i_t$ is a retraction on X for every $t \in I$ (see [7], p. 31;

cf. [19], p. 370). It is known ([9], Corollary, p. 93) that a dendroid
is contractible under a retracting homotopy if and only if it is smooth.
Smoothness being a hereditary property ([8], Corollary 6, p. 299) we
conclude by Proposition 1 (cf. [10], Proposition 14) that

Proposition 6. If a curve is contractible under a retracting homo-
topy, then it is hereditarily contractible.

The inverse is not true, as it can be easily seen from an example
defined as the disjoint union of two harmonic fans with the unique ac-
cumulation points on their bases joined by an arc otherwise missing
the fans. However if the curve under consideration is a fan, then —
using a rather complicated concept of an R-arc (see [10]) — one can
prove the converse, and get (see [10], Corollary 17)

Proposition 7. A fan is hereditarily contractible if and only if
it is smooth.

For arbitrary curves such an internal characterization of heredi-
tarily contractible ones is not known.

Let us call a dendroid X to be weakly non-contractible if X is not
contractible (in itself) and if there is a dendroid Y and a homeomor-
phism h: $X \longrightarrow h(X) \subset Y$ such that Y is contractible. Further, let us
call a dendroid X to be strongly non-contractible if X is not contr-
actible but it is not weakly non-contractible, i.e., if X cannot be
embedded into a contractible dendroid. Both classes are non-empty.
First, each contractible and non-smooth fan Y contains a (weakly) non-
-contractible fan X by Proposition 7; an example of a weakly non-contr-
actible dendroid has been described in [13]. Second, it is easy to see
by the heredity of uniform arcwise connectedness of dendroids that

Proposition 8. If a dendroid is not uniformly arcwise connected,
then it is strongly non-contractible.

Examples of not uniformly arcwise connected fans have been consider-
ed in [5], p. 199-202. An open problem is to give an internal charact-
erization of the above mentioned classes of non-contractible dendroids.

Suppose a curve (or, equivalently, a dendroid) X contains a point
p and a convergent point sequence $\{a_n\}$ such that the sequence of arcs
$\{pa_n\}$ is convergent. Denote by K the limit continuum $\text{Lim } pa_n$. The fol-
lowing problem seems to be interesting and worth to be studied: which
properties of K imply non-contractibility of X? It is known that non-
-local connectedness of K is one of such properties (see [10], Lemma
15). In other words, if X is contractible, then K must be a dendrite.
The problem is to specify the structure of the dendrite K.

In connection with this let us recall the following concept due to
Ralph B. Bennett [3]. A point p of a dendroid X is called a Q-point if
there exists in X a point sequence $\{p_n\}$ such that (i) $\{p_n\}$ converges

to p, (ii) the arcs pp_n converge to a non-degenerate limit continuum K, and (iii) if $K \cap pp_n = pq_n$, then the point sequence $\{q_n\}$ converges to p. It is an unproved conjecture that if a dendroid contains a Q-point, then it is not contractible. Since a Q-point of a subdendroid is obviously a Q-point of the whole dendroid, the positive answer to the above conjecture implies that every dendroid containing a Q-point is strongly non-contractible.

Some other known conditions of non-contractibility of continua are expressed in terms of the set function T. Given a compact Hausdorff space Y (not necessarily metric) and a set $A \subset Y$, we define T(A) as the set of all points y of Y such that every subcontinuum of Y which contains y in its interior must intersect A (see [11]). It is known (see e.g. [2], Corollary 1, p. 373) that if Y is a continuum and A is a subcontinuum of Y, then T(A) is a subcontinuum of Y. Recall that a mapping f: X $\longrightarrow$ Y of a topological space X into a topological space Y is said to be interior at a point $x_0 \in X$ provided that for every open set U containing x_0 in X, the point $f(x_0)$ is in the interior of f(U) in Y (see [20], p. 149). Using similar methods as in [18] one can prove the following.

Proposition 9. Let X be a compact Hausdorff space, Y be a Hausdorff continuum, A and B - closed subsets of Y such that $A \cap T(B) = \emptyset = B \cap T(A)$ and $T(A) \cap T(B) \neq \emptyset$, and let f: X $\longrightarrow$ Y be a continuous mapping of X into Y which is interior at a point $x_0 \in f^{-1}\big(T(A) \cap T(B)\big)$. If H: X $\times$ I $\longrightarrow$ Y is a homotopy with H(x,0) = f(x) for each $x \in X$, then $H(\{x_0\} \times I) \subset T(A) \cap T(B)$.

A proof of this proposition will be published somewhere. Taking in Proposition 9 X = Y and the identity for f and applying Proposition 3 we get Corollary 1 of [1], from which the next proposition, originally due to Bennett [3] and proved in [1], follows as a corollary.

Proposition 10. If X is a dendroid containing two points a and b such that $a \in X \setminus T(b)$, $b \in X \setminus T(a)$ and $T(a) \cap T(b) \neq \emptyset$, then X is not contractible.

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