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CATEGORIES OF CONTINUOUS FUNCTION SPACES

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A topological universal algebra E is said to be sufficiently complicated iff it is a Hausdorff algebra such that each character $C(X,E) \to E$ is an evaluation whenever X is an E-compact space. Then there is a categorical dual equivalence between the category TCP_E of all E-compact spaces and the category CF_E of all universal algebras $C(X,E)$ with X an arbitrary, not necessarily E-compact, topological space.

We obtain a new sufficiently complicated structure Z_∞ that will be used as an illuminating example in a general (though introductory!) study of the categories CF_E. This exposition has been influenced mainly by P. Brucker [1], [2], [3] and by P.R. Halmos [9]. Categorical notions not recalled in the text are taken from Z. Semandeni [10], Chapter III.

An object A of a category $\mathcal{A}$ is projective iff for each epimorphism $\alpha: B \to C$ and morphism $\beta: A \to C$ there is a morphism $\gamma: A \to B$ such that $\alpha\gamma = \beta$; injective objects are defined dually.

A Hausdorff space E is of compact regularity type iff there exists a compact space E_C such that the classes TCR_E and TCR_{E_C} of E-complete regularity and E_C-complete regularity respectively, are identical. Each zerodimensional space is of compact regularity type (E_C = finite discrete space), as is each completely regular space that contains a nonconstant continuous image of a real interval (E_C = a compact real interval). All sufficiently complicated structures in our knowledge have a compact regularity type.

1. The structure Z_∞

Let $Z_\infty = Z \cup \{\infty\}$ be the one-point compactification of Z; clearly Z_∞ is a zerodimensional compact Hausdorff space so that TCP_{Z_∞} coincides with TCP_{D_2} where D_2 is a two-point discrete space. Z_∞ will be provided with addition, multiplication and constant unary mapping onto 1, according to the following supplementary rules

*) "Aspirant" of the Belgian "Nationaal Fonds voor Wetenschappelijk Onderzoek"

$z+\infty=\infty+z=\infty+\infty=\infty$ for all $z\in\mathbb{Z}$

$z.\infty=\infty.z=\infty.\infty=\infty$ for all $z\in\mathbb{Z}$, $z\neq 0$

$0.\infty=\infty.0=0$

These operations are continuous and $\{0,\infty\}_{+,.}$ is isomorphic to the two-point discrete lattice $D_2=\{0,1\}_{\vee,\wedge}$. An elementary proof now shows that $\mathbb{Z}_\infty$ is sufficiently complicated; in fact we obtain a little more :

Theorem 1 : Let $X\in TCP_{\mathbb{Z}_\infty}$ and D a subset of $C(X,\mathbb{Z}_\infty)$ that satisfies

(i) All characteristic functions χ_U of clopen (open-and-closed) subsets U of X belong to D.

(ii) D is closed under + and . .

(iii) If $f,g\in D$ and $g(X)\subseteq\mathbb{Z}$, then there is a $k\in D$ with $f=g+k$.

Let $H:D\to\mathbb{Z}_\infty$ be a morphism for + and . such that $H(D)\not\subseteq\{0,\infty\}$; then H is the evaluation in a point of X.

Proof : (1) There is a point $x_0\in X$ such that $f(x_0)=0$ whenever $f\in D$ and $H(f)=0$. Indeed, otherwise we could find for each $x\in X$ a neighborhood U_x of x and a $f_x\in D$ such that $H(f_x)=0$ and f_x differs from 0 on U_x. Let $x_1,\dots x_n$ be chosen so that $X\subseteq U_{x_1}\cup\dots\cup U_{x_n}$ and set $f=f_{x_1}^2+\dots f_{x_n}^2$, then $H(f)=0$ and $f(x)\neq 0$ for all $x\in X$. If $g\in D$ is arbitrary, then $0=H(f)=H(f).H(\infty)=H(f.\infty)=H(\infty)=H(g+\infty)=H(g)+H(\infty)=H(g)$, so that H would be identically zero.

(2) $H(z)=z$ for all $z\in\mathbb{Z}$. Indeed, choose g such that $H(g)\notin\{0,\infty\}$, then from $H(g)=H(1).H(g)$ we infer $H(1)=1$; the general result follows from additivity properties of H.

(3) Whenever $f\in D$ and $H(f)\in\mathbb{Z}$, then $f(x_0)=H(f)$. By (iii) we can namely choose $g\in D$ such that $f=g+H(f)$ so that $H(f)=H(g)+H(H(f))=H(g)+H(f)$ by (2). Since $H(g)=0$ and thus $g(x_0)=0$ we obtain $f(x_0)=H(f)$.

(4) If $f\in D$ and $H(f)=\infty$, then there is an $x\in X$ with $f(x)=\infty$. Suppose $H(f)=\infty$, $f(X)\subseteq\mathbb{Z}$. Then we may find $z_1,\dots z_n\in\mathbb{Z}$ and clopen subsets $U_1,\dots U_n$ in X with $f=\chi_{U_1}.z_1+\dots+\chi_{U_n}z_n$. Let $i\in\{1,\dots n\}$ be such that $H(\chi_{U_i})=\infty$. Then $1=H(1)=H(\chi_{U_i})+H(\chi_{X\setminus U_i})=\infty$, a contradiction.

(5) If $f\in D$, then $H(f)=f(x_0)$. By (3) we may assume $H(f)=\infty$. Suppose $x_0\notin S=\{x\in X:f(x)=\infty\}$. There is a clopen $U\subseteq X$ such that $x_0\in U$ and $U\cap S=\emptyset$. Then $H(\infty.\chi_U)=\infty$ by (3) while $\infty=H(f.\chi_U)+H(f.\chi_{X\setminus U})$ so that $H(f.\chi_{X\setminus U})=\infty$ by (4). So $\infty=\infty.\infty=H(\infty.\chi_U).H(f.\chi_{X\setminus U})=H(0)=0$, which is again a contradiction.

Proposition 1 : $\mathbb{Z}_\infty$ is a retract of 2^N $(N=\{1,2,3,\dots\})$

Proof: For convenience we replace $\mathbb{Z}_\infty$ by its homeomorphic copy

$N_\infty=\{1,2,3,\ldots\}\cup\{\infty\}$. Mappings $f:N_\infty\to 2^N$ and $g:2^N\to N_\infty$ are defined by

$$[f(i)](j)=0 \text{ whenever } i<\infty \text{ and } j\neq i$$
$$[f(i)](j)=1 \text{ whenever } i<\infty \text{ and } j=i$$
$$[f(\infty)](j)=0 \text{ for all } j$$

and $g(a)=i<\infty$ whenever $a(j)=0$ for $j<i$ and $a(i)=1$

$$g(a)=\infty \text{ whenever } a(j)=0 \text{ for all } j$$

Then f and g are continuous and $g\circ f=\mathbf{1}_{N_\infty}$.

2. General properties of CF_E

Theorem 2 : If E is sufficiently complicated, then CF_E is complete and cocomplete.Furthermore, for each cardinal number m there is an m-free object, namely $C(E^m,E)$;the projections form a set of free generators. In CF_E each object is the epimorphic image of a free object.

Proofs : It may be shown without difficulty that TCP_E is complete and cocomplete (the completeness is very trivial) so that by duality CF_E has the same properties. The second assertion is easily verified (cfr. also P. Brucker [3], 4.1); the third is an immediate consequence.

Theorem 3 : If E is sufficiently complicated, then in CF_E each monomorphism is one-to-one. Furthermore, the following conditions are equivalent.

(α) Each epimorphism in CF_E is onto

(β) E is an injective object in TCP_E

(γ) Conditions (*) and (**) hold :

(*) : If $(X,t)\in TCP_E$, $(X,u)\in TCR_E$, $t\geq u$, then $t=u$ (where (X,t) is the set X, provided with topology t)

(**): If $A\subseteq B$ and $A,B\in TCP_E$, then A is E-embedded in B.

Proofs : Since CF_E has a 1-free object, each monomorphism in CF_E is one-to-one. A routine inspection will show the equivalence of (α), (β), (γ).

Lemma 1 : Let E be a T_2-space. Suppose $(X,t)\in TCR_E$ has the property

(P) Whenever $(X,u)\in TCR_E$ and $u\leq t$, then $u=t$

Then (X,t) is closed in each embedding in a E-completely regular space.

Proof : Let $(Y,u')\in TCR_E$ and $\phi:(X,t)\to(Y,u')$ determine a homeomorphism of (X,t) with $(\phi(X),u)$ where u is the relative topology of u'. If $\phi(X)$ is not closed in (Y,u') we choose $x_0\in\overline{\phi(X)}\setminus\phi(X)$ and $y_0\in\phi(X)$ (X may be assumed nonempty). Let u'_0 be the weak topology induced on Y by all $f_0 \in C((Y,u'),E)$ that are equal in x_0 and y_0 and let u_0 be the relative topology on $\phi(X)$. Then $(\phi(X),u_0)\in TCR_E$ and $u_0 \leqslant u$. The proof is now completed by showing $u_0 \neq u$.

Let $f\in C((Y,u'),E)$ be such that $f(x_0)\neq f(y_0)$; U_0 and U'_0 will denote disjoint open neighborhoods of $f(y_0)$ and $f(x_0)$ respectively; so $f^{-1}(U_0)$ and $f^{-1}(U'_0)$ are disjoint open u'-neighborhoods of y_0 and x_0; hence y_0 does not belong to the u-closure of $f^{-1}(U'_0)\cap\phi(X)$. On the other hand y_0 clearly belongs to the u_0-closure of that set, so $u\neq u_0$.

Lemma 2 : Let E be a T_2-space of compact regularity type, $(X,t)\in TCR_E$. The following are equivalent :

(1) If $(X,u)\in TCR_E$ and $u\leqslant t$, then $u=t$

(2) X is closed in each embedding in an E-completely regular space

(3) X is compact

Proofs : Lemma 1 gives (1)⇒(2). Since X may be homeomorphically embedded in a compact, E-completely regular space, (2)⇒(3) holds true. Finally (3)⇒(1) is obvious.

Proposition 2 : Let E be sufficiently complicated. If E is compact, then condition (*) holds. Conversely, if it holds, then E is at least countably compact. If E has a compact regularity type, then (*) holds if and only if E is compact.

Proofs : The first assertion is obvious. On the other hand, if E is not countably compact, then it contains a countable infinite discrete closed subset, so that $\mathbb{Z}\in TCP_E$. Also, $\mathbb{Z}_\infty \in TCP_{D_2}\subseteq TCP_E$. Since $\mathbb{Z}$ is not compact, a contradiction with (*) arises from the existence of a one-to-one mapping from $\mathbb{Z}$ onto $\mathbb{Z}_\infty$.
Finally, the last part of proposition 2 follows from lemma 2.

In view of proposition 2, it is natural to ask whether (**) holds for each compact space E. As a counterexample, set $E=[0,1]\cup[2,3]$ (usual topology), $A=\{0,1\}$, $B=[0,1]$. Then both A, B are E-compact and A is not E-embedded in B. (A similar situation always occurs when E is neither connected nor totally disconnected!). Nevertheless, most interesting sufficiently complicated compact algebras satisfy (**).
We need a simple categorical fact.

Proposition 3 : Let A,B,C,D be topological spaces, $A \subseteq B$, D a retract of C. If A is C-embedded in B, then A is D-embedded in B. (proof obvious)

Theorem 4 : If E is one of the structures $I=[0,1]$, $D_2=\{0,1\}$ or Z_∞, then each epimorphism in CF_E is onto.

Proofs : From [4], 3.11(c) we know that a compact subset of a completely regular space is R-embedded in it. Since I is a retract of R, the result holds in case $[0,1]$. It is easily seen that each compact subset of a zerodimensional compact space is D_2-embedded in it; this establishes the case D_2. The case Z_∞ now follows from the preceding one and propositions 1 and 3.

3. Injective and projective elements in CF_E

Proposition 4 : Let E be an arbitrary Hausdorff space. Then in TCP_E all monomorphisms are one-to-one. If we consider the assertions

(α) : E is an ss-space

(β) : Each epimorphism in TCP_E is onto a dense subset of its codomain

(γ) : Each E-compact space is E-maximal

Then (α) implies (β) and (β) implies (γ).

Proof : The notion of an ss-space is discussed in [8]; for E-maximal spaces we refer to [7]. Since TCP_E contains 1-free objects, all monomorphisms are one-to-one. Now

$(\alpha)\Rightarrow(\beta)$: If $A,B\in TCP_E$, $f\in C(A,B)$, $\overline{f(A)}\neq B$ then for $x\in B\setminus\overline{f(A)}$ we may find $n\geqslant 1$, $g\in C(B,E^n)$ and open $G\subseteq E^n$ such that $g(x)\in G$ and $g(\overline{f(A)})\cap G=\emptyset$.

By hypothesis there are ϕ, $\psi\in C(E^n,E)$ that are equal on $g(\overline{f(A)})$ but differ in $g(x)$ so that $\phi\circ g\circ f$ equals $\psi\circ g\circ f$ though $\phi\circ g$ differs from $\psi\circ g$.

$(\beta)\Rightarrow(\gamma)$: Let $A\in TCP_E$, A_0 its E-maximal extension, $i:A\to A_0$ the natural embedding. Then i is an epimorphism, so that $\overline{A}=A_0$; this is possible only if $A=A_0$.

Theorem 5 : If E is a sufficiently complicated space, then the singleton element is an injective object in CF_E. Furthermore :

(a) If there is a nonsingleton injective object in CF_E, then E is countably compact (a)' If E is of compact regularity type and there is such an object in CF_E, then E is compact

(b) Conversely, if E is a compact ss-space, then CF_E contains nonsingleton injective objects.

(c) If E is compact and there is at least one nonsingleton injective object in CF_E, then the injective objects are just the structures $C(X,E)$ where X is an extremely disconnected compact Hausdorff space.

Proofs : (a) If there is a nonsingleton injective object in CF_E, then each epimorphism in TCP_E clearly needs to be onto. Now, if E is not countably compact, then $\mathbb{Z}$ is E-compact. But $\mathbb{Z}$ is a dense subspace of its one point-compactification that, too, belongs to TCP_E.

(a') : This follows from lemma 2, part (2)⇒(3)

(b) : If E is a compact ss-space, then each epimorphism in TCP_E is onto. An application of [5], theorem 2.5 completes the proof.

(c) : Again, each epimorphism in TCP_E is onto, so that the result follows from [5], theorems 1.2 and 2.5.

The projective objects in CF_E are not easily characterizable. We know, however, some partial results :

Theorem 6 : If E is sufficiently complicated, then in CF_E

(a) A projective object is a retract of a free object

(b) A free object with at least one generator is projective iff each epimorphism in CF_E is onto; if so, all free objects are projective and the projective objects are just the retracts of the free objects.

(c) The free object without generators is projective.
If E is not countably compact, it is the only projective object.

(d) Even if the 0-free object is the only free projective object, there may be other projective objects

Proofs : (a) From theorem 2 we know that each object is the epimorphic image of a free object. By projectivity, it is a retract.

(b) From theorem 3 we know that each epimorphism in CF_E is onto iff E is injective in TCP_E. Now for each $m>0$, E is injective iff E^m is injective. Furthermore, a retract of a projective object is projective.

(c) The first assertion is obvious. If E is not countably compact, $\mathbb{Z}$ and $\mathbb{Z}_\infty$ both belong to TCP_E and $\mathbb{Z}$ is a dense subset of $\mathbb{Z}_\infty$, not D_2-embedded in it; since D_2 is a closed subspace of each nonempty nonsingleton element of TCP_E the result follows.

(d) As for an example, set $E=[0,1]\cup[2,3]$. From the remarks preceding proposition 3 we know that not every epimorphism in CF_E is onto; so the 0-free object is the only free projective object. On the other hand, $[0,1]$ belongs to TCP_E and is an injective object of that category. (this example is somewhat artificial since as far as we know $[0,1]\cup[2,3]$ has no interesting sufficiently complicated structure; from a purely theoretical point of view however, such a structure is definable; cfr. [8])

References

1. P. Brucker : Verbände stetiger Funktionen und kettenwertige Homomorphismen, Math.Ann. 187 (1970) 77-84
2. ———— : Eine Charakterisierung K-kompakter topologischer Räume, Monatshefte für Mathematik 75 (1971) 14-25
3. ———— : Dualität zwischen Kategorien topologischer Räume und Kategorien von K-Verbänden, ibid., 76 (1972) 385-397
4. L. Gillman and M. Jerison : Rings of continuous Functions, Van Nostrand 1960
5. A.M. Gleason : Projective topological Spaces, Ill.J.Math.2 (1958) 482-489
6. W. Govaerts : Representation and Determination Problems : A Case Study, Bull.Acad.Pol.Sci.Sér.Sci.Math.Astr.Phys. 24 (1976) 57-59
7. ———— : A modified Notion of E-compactness, ibid 61-64
8. ———— : A Separation Axiom for the Study of Function Space Structures, ibid. 65-69
9. P.R. Halmos : Lectures on Boolean Algebras, Springer Verlag, New York 1963 and 1974
10. Z. Semadeni : Banach Spaces of continuous Functions (vol.I), Warszawa 1971

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