Petr Kratochvíl
Multisequences and measures

In: Josef Novák (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part B: Contributed Papers. Society of Czechoslovak Mathematicians and Physicist, Praha, 1977. pp. 237--244.

Persistent URL: http://dml.cz/dmlcz/700679

MULTISEQUENCES AND MEASURES

P. KRATOCHVÍL

PRAHA

1. Introduction. Recall that a sequential convergence space is a nonempty set X provided with a sequential convergence. i. e. certain sequences in the set X are said to be convergent and each convergent sequence possesses a unique limit. Constant sequences are convergent and if a sequence converges to a point, then each subsequence of it converges to the same point. The closure λZ of a subset Z of X is the set of all limits of sequences in Z. The power of λ is defined recurrently, $\lambda^0 Z = Z$ and $\lambda^\xi Z = \bigcup_{\eta<\xi} \lambda(\lambda^\eta Z)$ for an ordinal number $\xi > 0$. If $x \in \lambda^{\omega_1} Z$, then there is the smallest ξ such that $x \in \lambda^\xi Z$. This ordinal number ξ is called the order of x with respect to Z. Evidently the order cannot be a limit ordinal number.

Sequential convergence spaces were investigated by J. Novak, who studied the measure extension problem [6]. He has reformulated an old idea due to Borel [1]: If Z is a set algebra with a probability measure, then each $\lambda^\xi Z$ is an algebra and $\lambda^{\omega_1} Z$ is the least σ-algebra. The measure is then extended successively to λZ, $\lambda^2 Z$, ..., and the transfinite induction is used. However, the outer measure is used. In this paper, we introduce a notion of a multisequence, which is a generalization of a double sequence to higher multiplicity. We attempt to eliminate the outer measure from the mentioned construction, thus completing the solution of J. Novak.

Borel's idea has been discussed by Luzin, who affirmed that "this idea is as natural as possible" [4]. It is still of interest even for contemporary mathematicians (see e. g. [2]). Convergence has been used in measure extension problems by Savel'jev (see his latest paper [7]).

Acknowledgement. I wish to express my gratitude to prof. J. Novák, who called my attention to extension problems.

2. Indices.

Denote $N := \{0, 1, 2, \dots\}$ the set of all natural numbers, $S = \bigcup \{N^n;\ n \in N\}$ the set of all finite sequences of natural numbers and $\Sigma = N^N$ the topological power of the topological discrete space N to the abstract set N. The space Σ is homeomorphic to the space of all irrational numbers. For $n \in N$, denote projections $pr_n : \Sigma \to N$, $\pi_n : \Sigma \to N^n$ and a shift $\tau_n : \Sigma \to \Sigma$. Given an element $\alpha = \langle i_0, i_1, \dots \rangle \in \Sigma$, the values are $pr_n(\alpha) := i_n$, $\pi_n(\alpha) := \langle i_0, i_1, \dots, i_{n-1}\rangle \in N^n$, and $\tau_n(\alpha) := \langle j_0, j_1, j_2, \dots\rangle$, where $j_0 = n$ and $j_{k+1} := i_k$ for $k \in N$. In a special case, $\pi_0(\alpha) = \langle\ \rangle = 0 \in N^0$ for all $\alpha \in \Sigma$, the value of $\pi_0(\alpha)$ being the only element of the power $N^0 = \{0\}$. All the defined projections and shifts are continuous. A neighborhood basis of a point $\alpha \in \Sigma$ consists of sets of a form $\pi_n^{-1}[\pi_n(\alpha)]$, $n \in N$.

Remark 2.1. (1) The finite subset $n = \{0, 1, \dots, n-1\} \subset N$ is compact. The theorem of Tychonoff implies compactness of a power $C_n = n^N \subset N^N = \Sigma$. Therefore the union $C = \bigcup\{C_n;\ n \in N\}$ is a σ-compact subset of Σ.

(2) Mappings of S into N are called edges and mappings of $S' = \bigcup\{N^n;\ n = 1, 2, \dots\} = S - \{0\}$ into N are called incomplete edges. For an incomplete edge g and a number $n \in N$, denote by $g^n : S \to N$ an edge that is defined by $g^n(0) = n$ and $g^n(s) = g(s)$ for $s \in S'$. For edges f_j and incomplete edges g_j, $j = 1, 2$, we shall write $f_1 \leq f_2$ iff $f_1(s) \leq f_2(s)$ for all $s \in S$ and $g_1 \leq g_2$ iff $g_1(s) \leq g_2(s)$ for all but finite number of $s \in S'$. If $g_1 \leq g_2$ and $g_2 \leq g_1$, then g_1 and g_2 are said to be equivalent. Thus, the set of all edges is a partially ordered one and the set of all incomplete edges is a preordered one. While the former set is directed, each countable subset the latter set has an upper bound. Moreover, the combinatorial lemma of Martin and Solovay [5] implies the following modification of a result of S. Hechler and K. Kunen.

Proposition 2.1. Assume Martin's axiom to be true, let $\varkappa < c := 2^{\aleph_0}$ and let $\{g_\iota;\ \iota < \varkappa\}$ be a family of incomplete edges. Then there is an upper bound $g \in N^{S'}$, $g_\iota \leq g$ for each $\iota < \varkappa$.

Corollary 2.1. There is a cofinal in $N^{S'}$ well-ordered chain of incomplete edges $\{g_\iota;\ \iota < c\}$, $g_i < g_j$ for $i < j < c$ and given an incomplete edge g there is $\iota < c$ such that $g \leq g_\iota$.

Proof. The Corollary 2.1 is an easy consequence of Lemma 2.1 and the principle of the transfinite construction.

For an edge f, denote a subset Σ_f of Σ by

$$(1)\quad \Sigma_f = \{\alpha;\ \alpha \in \Sigma \text{ and } f(\pi_n\alpha) \leq pr_n(\alpha) \text{ for each } n \in N\}.$$

The set Σ_f is called a section. Given f, define $i_0 = f(0)$ and $i_n = f(\langle i_0, i_1, \dots, i_{n-1}\rangle)$. We see that $\langle i_0, i_1, \dots\rangle \in \Sigma_f$, i. e. $\Sigma_f \neq 0$ for every edge f.

Lemma 2.1. The family $\{\Sigma_f;\ f \in N^S\}$ is a basis of a filter in Σ. This filter is called a filter of sections.

We omit the easy proof of Lemma 2.1.

Lemma 2.2. Let g, h be incomplete edges and let $h \leq g$. Then $\Sigma_{g^m} \subset \Sigma_{h^m}$ for all but a finite number of $m \in N$.

Proof. The relation $g(s) < h(s)$ is true only for finite number of $s \in S'$. Let $s_1, s_2, \ldots, s_j$ denote all the exceptional elements of S', $s_r = \langle i_0^r, i_1^r, \ldots, i_{k_r}^r \rangle$, $r = 1,2,\ldots,j$. Denote $m_0 = \max\{i_0^r; r=1,2,..,j\}$. According to (1), $\alpha = \langle i_0, i_1, \ldots \rangle \in \Sigma_{g^m}$ implies $i_0 = pr_0(\alpha) \geq g^m(\pi_0 \alpha) = g^m(0) = m$. Therefore $\pi_n(\alpha) = \langle i_0, i_1, \ldots, i_{n-1} \rangle$ is not equal to any exceptional element of S' if $m > m_0$, i. e. $h(\pi_n \alpha) \leq g(\pi_n \alpha)$ for $\alpha \in \Sigma_{g^m}$ and $m > m_0$. Suppose $m > m_0$ and $\alpha \in \Sigma_{g^m}$, i. e. the condition in (1) is fulfilled for $f = g^m$. Since $h^m(0) = m = g^m(0)$, the condition in (1) is fulfilled for $f = h^m$ and the proof is finished.

Corollary 2.2. If incomplete edges g, h are equivalent, then $\Sigma_{g^m} = \Sigma_{h^m}$ for all but a finite number of $m \in N$.

3. Multisequences.

Definition 3.1. Let X be a set and let $\Theta : \Sigma \to X$. The mapping Θ is said to be a multisequence in the set X if Θ is a continuous mapping of Σ into the set X provided with the discrete topology.

Remark 3.1. Recall that $\Theta(\alpha)$ is an isolated point of the discrete space X for each $\alpha \in \Sigma$. The continuity of Θ implies the existence of a neighborhood $U_n = \pi_n^{-1}[\pi_n \alpha]$ of α such that $\Theta[U_n] = \{\Theta(\alpha); \alpha \in U_n\}$ is contained in the open set $\{\Theta(\alpha)\}$, i. e. $\Theta[U_n] = \{\Theta(\alpha)\}$.

Definition 3.2. Let $h(\alpha)$ be the least natural number n such that $\Theta[U_n] = \{\Theta(\alpha)\}$. The mapping $h : \Sigma \to N$ is called the multiplicity function of the multisequence Θ.

Proposition 3.1. A multiplicity function h of a multisequence Θ is a multisequence in N.

Proof. It is easy to verify that $h[U_n] = \{n\} = \{h(\alpha)\}$ for $n = h(\alpha)$ and $U_n = \pi_n^{-1}[\pi_n \alpha]$, which proves the Proposition 3.1.

Proposition 3.2. The multiplicity function h' of a multiplicity function h is less than or equal to h.

Proof. Similarly as in the proof of Proposition 3.1., if $h[U_m] = \{n\}$ holds for $m < n$, then $h'(\alpha) < h(\alpha)$. In the opposite case, $h'(\alpha) = h(\alpha)$.

Example 3.1. Let $x_{mn} = x_s$, $s = \langle m,n \rangle \in N^2$, be a double sequence in a set X. Define $\Theta(\alpha) = x(\pi_2 \alpha) = x_{i_0 i_1}$ for each $\alpha = \langle i_0, i_1, \ldots \rangle \in \Sigma$. The mapping Θ is constant on each neighborhood $U_2 = \pi_2^{-1}(\pi_2 \alpha)$, hence it is a multisequence in X. Clearly $\Theta[U_2] = \{x(\pi_2 \alpha)\}$ and $\Theta[\pi_1^{-1}(\pi_1 \alpha)] = \{x_{mn}; m = \pi_1 \alpha, n \in N\} = \{x_{i_0 n}; n \in N\}$. Therefore the multiplicity function $h(\alpha) = 2$ for all $\alpha \in \Sigma$ if all the points x_{mn} are distinct. The multiplicity function of h is identically zero as $h[\pi_0^{-1}(\pi_0\alpha)] = h[\Sigma] = \{2\}$.

Remark 3.2. Similarly, we can derive a multisequence starting from a simple sequence $\langle y_n \rangle$. We identify the sequence and the derived multisequence, e. g. $\langle x_{mn} \rangle \equiv \Theta$, and we use a phrase " a double sequence Θ ", etc.

Example 3.2. Let Z be a subset of a sequential convergence space X (e. g. X = the set of al real functions with pointwise convergence and Z = the subset of all continuous functions) and let $x \in \lambda^{\omega_1} Z$ (e. g. x = a Baire measurable function). Denote the order of x with respect to Z in X by ξ, $x \in \lambda^{\xi} Z$, $\xi < \omega_1$. If $\xi = 0$, we define $x_m = x$, $\xi_m = 0$ for all $m \in N$. If $\xi > 0$, then there are points $x_m \in \lambda^{\xi-1} Z$, $m \in N$, such that $x = \lim x_m$. Denote ξ_m the order of x_m with respect to Z, hence $\xi_m < \xi$ for all $m \in N$.

Now, let m be a fixed number. If $\xi_m = 0$, then define $x_{mn} = x_m$, $\xi_{mn} = 0$ for each $n \in N$. If $\xi_m > 0$, then there are points x_{mn}, $n \in N$, such that $x_m = \lim x_{mn}$ as $n \to \infty$. Denote ξ_{mn} the order of x_{mn}, hence $\xi_{mn} < \xi_m$ for $n \in N$ and $m \in N$. Write i_0 instead of m and i_1 instead of n.

Denote ξ_s the order of x_s with respect to Z and suppose that x_s has been defined for each $s \in \bigcup\{N^r;\ r=0,1,\ldots,k\}$ in such a way that for each $r = 0,1,\ldots,k-1$ and $s = \langle i_0, i_1, \ldots, i_{r-1} \rangle$

(1) $\quad x_{i_0,i_1,\ldots,i_{r-1}} = \lim\limits_{i_r \to \infty} x_{i_0,i_1,\ldots,i_r}$. and

(2) $\quad \xi \geq \xi_1 \geq \xi_{i_1 i_2} \geq \ldots \geq \xi_{i_0,i_1,\ldots,i_{k-1}}$

for each $s = \langle i_0 . i_1 . \ldots , i_{k-1} \rangle \in N^k$, and

(3) any two ξ's in (2) can be equal only if they are both zero.

Now, let $s = \langle i_0, i_1, \ldots, i_{k-1} \rangle \in N^k$ be given. If $\xi_s = 0$, then we define $x_{i_0,i_1,\ldots,i_k} = x_s$ and $\xi_{i_0,i_1,\ldots,i_k} = \xi_s$ for all $i_k \in N$. If $\xi_s > 0$, then there are $x_{i_0,i_1,\ldots,i_k}$ with orders with respect to Z $\xi_{i_0,i_1,\ldots,i_k} < \xi_s$ such that $x_s = \lim\limits_{i_k \to \infty} x_{i_0,i_1,\ldots,i_k}$. Thus x_s and ξ_s are defined for all $s \in S$ and (1), (2) and (3) are fulfilled for all $k \in N$ and $s \in S$.

Given $\alpha = \langle i_0, i_1, \ldots \rangle \in \Sigma$, the sequence $\xi \geq \xi_{i_0} \geq \xi_{i_0,i_1} \geq \ldots$ of ordinal numbers cannot be strongly descending. Denote $k \in N^1$ the least number such that $\xi_{i_0,i_1,\ldots,i_{k-1}} = 0$. Denote $\Theta(\alpha) = x_{i_0,i_1,\ldots,i_{k-1}}$ and $h(\alpha) = k$. Evidently Θ is a multisequence in Z and h is a multiplicity function of Θ.

Proposition 3.3. If Θ is a multisequence and C the set in Remark 2.1, then $\Theta[\Sigma] = \Theta[C]$. Therefore the complete Θ-image of Σ, $\Theta[\Sigma]$, is a countable set.

Proof. Let $\alpha = \langle i_0, i_1, \ldots\rangle \in \Sigma$ and the multiplicity function of Θ at α be $h(\alpha) = k$. Define $j_r := i_r$ or $j_r := i_k$ according to whether $r < k$ or not. Denote $\beta = \langle j_0, j_1, \ldots\rangle$, $n = \max\{i_0, i_1, \ldots, i_k\}$. Then $\Theta(\alpha) = \Theta(\beta)$ and $\beta \in C_{n+1} \subset C$. Since continuous images of compact sets C_n are finite, the proof is finished.

Definition 3.3. Let a multisequence Θ in X and a natural number n be given. Define $\Theta^n(\alpha) = \Theta(\tau_n\alpha)$ for all $\alpha \in \Sigma$. The mapping Θ^n is a multisequence in X. It is called a restriction of Θ to n.

Definition 3.4. Let M be the set of all multisequences in a set X. We use the transfinite induction.

(i) define $M_0 = \{\Theta; \Theta \in M$ and Θ is a constant mapping$\}$ and $M_\xi = 0$ if ξ is a limit ordinal number.

(ii) Let $\xi > 0$ be an ordinal number and let M_η be defined for each $\eta < \xi$. We shall write $\mu\Theta = \eta$ iff $\Theta \in M_\eta$.

Define $M_\xi = \{\Theta; \Theta \in M, \Theta^n \in \cup\{M_\eta; \eta < \xi\}, \xi = (\sup_{n\in N} \mu\Theta^n) + 1\}$. Thus the set M_ξ is defined for each countable ordinal number ξ. The ordinal number $\mu\Theta$ is called the multiplicity of Θ. The following lemma shows that $M = \cup\{M_\xi; \xi < \omega_1\}$.

Lemma 3.1. The multiplicity $\mu\Theta$ is defined for every multisequence Θ. It can be equal to any countable isolated ordinal number.

Proof. If $\mu\Theta$ is not defined, then there is $i_0 \in N$ such that $\mu\Theta^{i_0}$ is not defined, which implies the existence of $i_1 \in N$ such that $\mu(\Theta^{i_0})^{i_1}$ is not defined, etc. Denote $\alpha = \langle i_0, i_1, \ldots\rangle \in \Sigma$, $k = h(\alpha)$. Then $\mu((\ldots(\Theta^{i_0})^{i_1})\ldots)^{i_k} = 0$, which is a contradiction with the definition of $i_0, i_1, \ldots$.

The multiplicity of Θ in Example 3.2 is equal to ξ and the proof of the Lemma 3.1 is finished.

Example 3.3. Given $m \in N$, the restriction of the double sequence from Example 3.1 is a simple sequence $\langle y_n\rangle$, $y_n = x_{mn}$ for all $n \in N$. The multiplicity of a simple sequence is equal to 1 and that of a double sequence is equal to 2.

4. Multisequences in set algebras and the extension of probability measures.

In this section, we introduce a notion of convergence of multisequences. We show that a quasi-continuous mapping on a subset Z of X can be extended to $\lambda^{\omega_1} Z$. The existence of an extension of a measure is a natural consequence.

Definition 4.1. Let Θ be a multisequence in the set R of all real numbers. We define an upper limit

$$\limsup \Theta = \inf \{\sup \Theta[\Sigma_f];\ f \in N^S\},$$

and a lower limit $\liminf \Theta = -\limsup(-\Theta)$. A multisequence Θ in R is said to be convergent iff $\limsup \Theta = \liminf \Theta$ and this common value is called the limit of Θ and denoted by $\lim \Theta$.

We identify a subset A of a given set Ω with the characteristic function of A. If Θ is a multisequence in the set of all real functions, we define $\limsup \Theta$, $\liminf \Theta$ and $\lim \Theta$ pointwise. Thus convergence of multisequences in the algebra $\mathcal{P}(\Omega)$ of all subsets of Ω is introduced.

Proposition 4.1. Assume Martin's axiom holds and let Θ be a multisequence in a set algebra Z. Then there is a chain $\langle A_i;\ i<c\rangle$ of length $c := 2^{\aleph_0}$, $A_i \supset A_j$ for $i<j$, such that $A_i \in \lambda^2 Z$ for all $i<c$ and $\limsup \Theta = \bigcap \{A_i;\ i < c\}$.

Proof. If $g \in N^{S'}$ is an incomplete edge, recall that $g^n(0) = n$ and denote $A_g := \bigcap \{ \bigcup \Theta[\Sigma_{g^n}];\ n \in N\} \in \lambda^2 Z$. According to the Definition 4.1 $\limsup \Theta = \bigcap\{\bigcup \Theta[\Sigma_{g^n}];\ n \in N,\ g \in N^{S'}\} = \bigcap_g \bigcap_n (\bigcup \Theta[\Sigma_{g^n}]) = \bigcap \{A_g;\ g \in N^{S'}\}$. Corollary 2.1 implies the existence of a chain g_i, $i < c$, which is cofinal in the set of all incomplete edges. Since $A_g \supset A_h$ for $g \leqslant h$, the proof is finished.

If $\Theta : \Sigma \to Z$ is a multisequence and $P : Z \to [0, 1]$ is a mapping, then the composition $P\Theta : \Sigma \to [0, 1]$ of the two continuous mappings Θ and P is a continuous mapping into the interval $[0, 1]$ provided with the discrete topology.

Lemma 4.1. Let Θ be a multisequence in a set algebra Z and let P be a probability measure on Z. Then there is a sequence of compact subsets $K_n \subset \Sigma$, $n \in N$, such that for each number $\varepsilon > 0$ there is a subseqence $\langle L_j\rangle$ of $\langle K_n\rangle$, $n_1 < n_2 < \ldots$, $L_j = K_{n_j}$, such that

$$P\Big(\bigcap_{j=1}^{m} \bigcup \Theta[L_j]\Big) \geqslant \limsup P\Theta - \varepsilon \tag{4.1.1}$$

for each $m \in N$, and

(4.1.2) for each edge $f \in N^S$ there is $j \in N$ such that $\Theta[L_j] \subset \subset \Theta[\Sigma_f]$.

An outline of the proof.

(i) If the multiplicity of Θ is equal to 0, i. e. Θ is a constant mapping, choose an arbitrary element $\alpha \in \Theta$ and define $K_n = \{\alpha\}$ for all n.

(ii) Assertion. If $P(z-z_n) \leqslant \varepsilon \cdot 2^{-n-1}$, then $P(\bigcap_{n=1}^{m} z_n) \geqslant Pz - \varepsilon/2$. This assertion is a straightforward consequence of De Morgan rules, $z - \bigcap z_n = \bigcup (z - z_n)$. Let the multiplicity of Θ be equal to 1, i. e. we have a simple sequence $\langle x_n \rangle$ (see the Remark 3.2). Let $\varepsilon > 0$ be given. Put $A_n^m = \bigcup_{j=n}^{m} x_j$, $a_n = \sup \{PA_n^m;\ m \in N\}$ and $\eta = (\limsup Px_n - \varepsilon)/2$. There are m_n such that $a_n - PA_n^{m_n} < \varepsilon \cdot 2^{-n-1}$ for all n. Choose compact subsets $K_n \subset \Sigma$ such that $n \leqslant pr_0 \alpha \leqslant m_n$ for all $\alpha \in K_n$. Let $z_n := \bigcup \Theta[K_n]$. For sufficiently large k we have $Px_k > \limsup Px_n - \varepsilon/2$. Put $z = x_k$. The assertion implies that the Lemma 4.1 is true for $L_n = K_n$ and the given number ε. Now, if ε is changed, then put $L_j = K_{j+p}$ for a suitable p.

(iii) The general case. Let the multiplicity of Θ be equal to ξ, $1 < \xi < \omega_1$, and let the Lemma 4.1 hold for every multisequence with the multiplicity less than ξ. Thus for each restriction Θ^n, there is a family $\{K_{nm}\}$ fulfilling the conditions of the Lemma 4.1. Let $K_s^t := \tau_{n_1} K_{n_1 m_1} \cup \tau_{n_2} K_{n_2 m_2} \cup \dots \cup \tau_{n_k} K_{n_k m_k}$, $k \in N$, $s = \langle n_1, \dots, n_k \rangle \in S$, $t = \langle m_1, \dots, m_k \rangle \in S$.

Similarly as in (ii) it can be verified that $\{K_s^t;\ k \in N, s, t \in N^k\}$ is a countable family of compact subsets of Σ that satisfies the condition of Lemma 4.1.

<u>Theorem 4.1.</u> Let P be a probability measure on a set algebra Z and let Θ be a multisequence in Z such that $\limsup \Theta \in Z$. Then $\limsup P\Theta \leqslant P(\limsup \Theta)$.

<u>Proof.</u> Given $\varepsilon > 0$, let L_j be such that (4.1.1) and (4.1.2) holds, i. e. for each edge f there is j such that $A_j := \bigcup \Theta[L_j] \subset \bigcup_m \Theta[\Sigma_f] =: B_f$. Therefore $\bigcap_{j=1}^{\infty} A_j \subset \bigcap_f B_f = \limsup \Theta$ and $\lim_m (\bigcap_{j=1}^{m} A_j - \limsup \Theta) = 0$. The continuity of P implies the existence of m_0 such that $P(\bigcap_{j=1}^{m_0} A_j - \limsup \Theta) < \varepsilon$. Therefore $P(\limsup \Theta) \geqslant \limsup P\Theta - 2\varepsilon$ and the proof is finished.

<u>Lemma 4.2.</u> Let $P : Z \to Y$ be a mapping of a subset Z of a sequential convergence space X into a sequentially compact space Y. Then there is a mapping $Q : \lambda^{\omega_1} Z \to Y$, called a quasi-continuous extension of P, such that : (i) the restriction $Q_{|Z} = P$, and (ii) for each $x \in \lambda^{\omega_1} Z$ with an order $\xi > 0$ there are y_n with orders $\xi_n < \xi$, $n \in N$, such that $x = \lim y_n$, $Q(x) = \lim Q(y_n)$.

Once such sequences have been chosen, they are called special; for $\xi = 0$ we put $y_n := x$.

<u>Proof.</u> The mapping Q will be constructed by transfinite induction. For $\xi > 0$ we choose $x_n \in \lambda^{\xi-1} Z$, $x = \lim x_n$. The sequential compactness of Y implies the existence of a suitable subsequence

$<y_n>$ of $<x_n>$.

Theorem 4.2. Every probability measure on a set algebra Z can be extended to a probability measure Q on the σ-algebra $\lambda^{\omega_1}Z$.

Proof. Let Y be the interval [0, 1]. According to Lemma 4.2, there is a quasi-continuous extension Q of P. We shall prove the continuity of Q at 0. The other assertions of Theorem 4.2 are either easy to prove or they are proved in[6]. So assume Q not to be continuous at 0. There are $x_m \in \lambda^{\omega_1}Z$, $m \in N$, and $\varepsilon > 0$ such that $0 = \lim x_m$ and $Q(x_m) > \varepsilon$ for all m . For a fixed m and $x := x_m$, acording to (ii) of Lemma 4.2, there is a special sequence $<y_n>$. Put $x_{mn} := y_{n+1}$ for a suitable 1 such that $Q(x_{mn}) > \varepsilon$; of course $x_m = \lim x_{mn}$. Similarly as in Example 3.2, we construct a multisequence such that $\lim\sup \Theta = 0$ and $\Theta(\alpha) > \varepsilon$ for all $\alpha \in \Sigma$. The last relations imply $\lim\sup P\Theta \geq \varepsilon$, which is a contradiction to Theorem 4.1.

Remark 4.1. Notice that in the proofs of the main statements we have not used elements outside of the algebra Z , and that is what we mean by the phrase "not using the outer measure".

References.

[1] E. Borel: Leçons sur la théorie des fonctions. Gauthier-Villars, Paris 1914.

[2] L. W. Cohen: Measure and integration in the manner of Borel. Scripta Math. 29 (1973), 417-435.

[3] J. Dugundji: Topology. Allyn and Bacon, Inc., Boston, 1966.

[4] N. N. Lusin: Lekcii ob analitičeskih množestvah i ih priloženijah. Gos. izd. teh.-teor. lit., Moskva 1953.

[5] D. A. Martin, R. M. Solovay: Internal Cohen extensions. Ann. Math. Logic 2 (1970), 143-178.

[6] J. Novák: Über die eindeutigen stetigen Erweiterungen stetiger Funktionen. Czechoslovak Math. J. 8 (83) (1958), 344-355.

[7] L. Ja. Savel'ev: Teorema o prodolženii sekvencial'nyh mer. Dokl. Akad. Nauk SSSR 229 (1976), 307-309.