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EXTENSIONS OF CLOSURE SPACES AND APPLICATIONS TO PROXIMITY AND CONTIGUITY STRUCTURES

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1. Introduction. In the very infancy of general topology F. Riesz [4] asked: which types of proximities have the property that they can be induced by elementary proximities on suitably constructed extensions of the original space. In 1952 Smirnov [5] showed that EF - proximities have this property. In 1973 one of the present authors [7] proved that the much larger class of LO - proximities has the property. In both of these cases the underlying spaces as well as the extensions are topological spaces.

However, the closure operator induced by basic proximities is not in general a Kuratowski closure operator, but only satisfies the conditions C_1, C_2, C_3, given below. Partly because of this Čech [1] introduced the concept of a closure space. To investigate whether the property of Riesz holds for larger classes of proximity structures it becomes necessary to develop an extension theory of closure spaces. Except for definitions of some of the basic concepts in [3] and results on embedding of closure spaces in cubes in [1] and [8] this is the first time that the subject has been more thoroughly studied. A detailed exposition of our investigation is given in [2]. Here we restrict ourselves to presenting those results which will be used in the applications to proximity and contiguity structures. This is done in Section 2. The results are then employed in Section 3 to prove that all separated RI - proximities have the property of Riesz and that an analogous result holds for separated RI - contiguities.

A function $c: \mathfrak{P}(X) \to \mathfrak{P}(X)$ is called a <u>closure operator on</u> X if it satisfies the following three conditions:

C_1: $c(\phi) = \phi$

C_2: $c(A) \supset A$,

C_3: $c(A \cup B) = c(A) \cup c(B)$.

A pair (X,c) where c is a closure operator on the set X, is called a <u>closure space</u>. These concepts are generalizations of the more familiar Kuratowski closure operator and topological space, respectively.

In our development the concept of a grill plays an important role. A <u>grill on</u> X is a collection $\mathfrak{G}$ of subsets of X satisfying

G_1: $B \supset A \in \mathfrak{G} \Rightarrow B \in \mathfrak{G}$

G_2: $A \cup B \in \mathfrak{G} \Rightarrow A \in \mathfrak{G}$ or $B \in \mathfrak{G}$

G_3: $\phi \notin \mathfrak{G}$.

Dual to the concept of neighborhood filter of a point is that of adherence grill of a point x. By it we mean the grill

$$\mathfrak{G}_c(x) = [A\colon x \in c(A)] .$$

Since

$$c(A) = [x\colon A \in \mathfrak{G}_c(x)]$$

it is clear that knowledge of all $\mathfrak{G}_c(x)$, $x \in X$, determines c completely just as knowledge of c determines all $\mathfrak{G}_c(x)$. The following lemma is an easy consequence of the appropriate definitions.

Lemma 1. If (X,c) is a closure space then $\mathfrak{G}_c(x)$ is a grill on X, for all $x \in X$. If for each $x \in X$ the family $\mathfrak{G}_x$ is a grill on X containing $[x]$, then the operator $g\colon \mathfrak{P}(X) \to \mathfrak{P}(X)$ defined by

$$g(A) = [x\colon A \in \mathfrak{G}_x]$$

is a closure operator on X.

A closure space (X,c) shall be called a G_0 - space if

$$\mathfrak{G}_c(x_1) = \mathfrak{G}_c(x_2) \Rightarrow x_1 = x_2 .$$

In what follows there is always an underlying nonempty set X. It will be convenient to denote elements of X by $x,y,\ldots,$ subsets by $A,B,\ldots$. Families of subsets will be denoted by $\mathfrak{A},\mathfrak{B},\ldots$. In particular, $\mathfrak{U}$, $\mathfrak{V}$ will be used for ultrafilters and $\mathfrak{G}$ for grills. Letters $\alpha,\beta,\gamma,\ldots$ shall be used for collections of families of sets (i.e., $\alpha \subset \mathfrak{P}(\mathfrak{P}(X))$). There will be some exceptions to these conventions. The notation $|A|$, $|\mathfrak{A}|$, ... refers to the cardinal number of the set under consideration.

2. Extensions of closure spaces. Let $\psi\colon X \to Y$ be an injection, let c be a closure operator on X and k be a closure operator on Y then

$$E = (\psi, (Y,k))$$

is called an extension of (X,c) if

(1) $$\psi(c(A)) = k(\psi(A)) \cap \psi(X), \quad \text{for all } A \subset X,$$

and

(2) $$k(\psi(X)) = Y .$$

Since ψ is an injection (1) insures that ψ is a homeomorphism from (X,c) onto $(\psi(X),k')$, where $k'(B) = k(B) \cap \psi(X)$, for all $B \subset \psi(X)$, is the closure operator induced on $\psi(X)$ by the closure operator k on Y. Condition (2) insures that $\psi(X)$ is dense in (Y,k).

Associated with each extension is its dual trace system

$$X^* = X^*(E) = [\tau(y,E): y \in Y] ,$$

where

$$\tau(y) = \tau(y,E) = [A: y \in k(\psi(A))] .$$

We speak of dual trace systems X^* and dual traces $\tau(y)$ since the terms "trace system" and "trace" are usually reserved for the families of filters and individual filters, respectively, which are the traces on X of the neighborhood filters of y on Y .

The following lemma is easily established.

Lemma 2. (a) For all extensions E and all $y \in Y$ the trace $\tau(y,E)$ is a grill on X . (b) $\tau(\psi(x),E) = \mathfrak{G}_c(x)$.

We are now able to state and prove our main result.

Theorem 1. Let (X,c) be a given G_0 - closure space. Let X^* be a collection of grills on X satisfying

$$[\mathfrak{G}_c(x): x \in X] \subset X^* .$$

Define

$$A^* = [\mathfrak{G}: \mathfrak{G} \in X^* , A \in \mathfrak{G}] ,$$

$$\varphi: X \to X^* \text{ by } \varphi(x) = \mathfrak{G}_c(x)$$

$$h_r(\alpha) = (\varphi^{-1}(\alpha))^* \cup r(\alpha \sim \varphi(X)) , \alpha \subset X^* ,$$

where $r: \mathfrak{P}(X^* \sim \varphi(X)) \to \mathfrak{P}(X^*)$ satisfies

(3) $$r(\emptyset) = 0 , \quad r(\beta) \supset \beta , \quad r(\beta_1 \cup \beta_2) = r(\beta_1) \cup r(\beta_2) .$$

Then $(\varphi,(X^*,h_r))$ is a G_0 - extension of (X,c) with dual traces $\tau(\mathfrak{G}) = \mathfrak{G}$, for all $\mathfrak{G} \in X^*$.

Proof: Clearly φ is an injection into X^* . Moreover

$$h_r(\varphi(X)) = X^* \cup r(\emptyset) = X^*$$

so that $\varphi(X)$ is dense in X^* . Since $\tau(\mathfrak{G}) = [B: \mathfrak{G} \in h_r(\varphi(B))]$ the condition $\tau(\mathfrak{G}) = \mathfrak{G} = [B: B \in \mathfrak{G}]$ is equivalent to

$$B \in \mathfrak{G} \Leftrightarrow \mathfrak{G} \in h_r(\varphi(B)) .$$

This is the case iff

$$h_r(\varphi(B)) = B^* .$$

If h is any closure operator on X^* then it is additive and hence

$$h(\alpha) = h(\alpha \cap \varphi(X)) \cup h(\alpha \sim \varphi(X)) .$$

Now $\alpha \cap \varphi(X) = \varphi(\varphi^{-1}(\alpha))$ and hence the requirement $\tau(\mathfrak{G}) = \mathfrak{G}$ for all $\mathfrak{G} \in X^*$ is equivalent to

$$h(\alpha \cap \varphi(X)) = (\varphi^{-1}(\alpha))^* \quad \text{and} \quad h(\varphi(A) \sim \varphi(X)) = h(\phi) = \phi .$$

Thus we can write

$$h_r(\alpha) = (\varphi^{-1}(\alpha))^* \cup r(\alpha \sim \varphi(X)) ,$$

where r is defined on $\mathfrak{P}(X^* \sim \varphi(X))$ with values in $\mathfrak{P}(X^*)$ and is arbitrary except for satisfying the condition (3). We now note that

$$h_r(\varphi(A)) \cap \varphi(X) = A^* \cap \varphi(X)$$
$$= [\mathfrak{G}: \mathfrak{G} = \mathfrak{G}_c(x) , \ A \in \mathfrak{G}]$$
$$= [\mathfrak{G}_c(x): x \in c(A)] = \varphi(c(A)) .$$

Hence φ is a homeomorphism onto $(\varphi(X),h_r')$ where $h_r'(\beta) = h_r(\beta) \cap \varphi(X)$, and $(\varphi,(X^*,h_r))$ is an extension of (X,c) .

Note that for every $\mathfrak{G} \in X^*$

$$\mathfrak{G}_{h_r}(\mathfrak{G}) \cap \mathfrak{P}(\varphi(X)) = [\varphi(A): A \in \mathfrak{G}] .$$

Thus $\mathfrak{G} \neq \mathfrak{G}'$ implies $\mathfrak{G}_{h_r}(\mathfrak{G}) \neq \mathfrak{G}_{h_r}(\mathfrak{G}')$ and hence (X^*,h_r) is a G_0 - space.

3. Applications. We begin this section by reviewing a number of definitions for proximities and contiguities.

A collection Π of families of subsets of X is called a basic proximity on X if it satisfies the following requirements:

P_0: $\mathfrak{A} \in \Pi \Rightarrow |\mathfrak{A}| = 2$

P_1: $|\mathfrak{A}| = 2 , \ \cap[A: A \in \mathfrak{A}] \neq \phi \Rightarrow \mathfrak{A} \in \Pi$,

P_2: $\Pi(A) = [B: [A,B] \in \Pi]$ is a grill on X for all $A \subset X$.

There are other, equivalent formulations for a proximity we chose this one because it parallels the definition of a basic contiguity $\mathfrak{F}$ on X given below:

Con_0: $\mathfrak{A} \in \mathfrak{F} \Rightarrow |\mathfrak{A}| < \aleph_0$

Con_1: $|\mathfrak{A}| < \aleph_0 , \ \cap[A: A \in \mathfrak{A}] \neq \phi \Rightarrow \mathfrak{A} \in \mathfrak{F}$,

Con_2: $\mathfrak{F}(\mathfrak{A}) = [B: [B] \cup \mathfrak{A} \in \mathfrak{F}]$ is a grill for all $\mathfrak{A} \subset \mathfrak{P}(X)$,

Con_3: $\mathfrak{B} \subset \mathfrak{A} \in \mathfrak{F} \Rightarrow \mathfrak{B} \in \mathfrak{F}$.

A proximity (contiguity) is called separated if

$$[[x],[y]] \in \Pi \Rightarrow x = y ,$$
$$[[x],[y]] \in \mathfrak{F} \Rightarrow x = y .$$

A proximity (contiguity) is called a Riesz or RI - proximity (contiguity) if

$[A,[x]] \in \Pi$ and $[B,[x]] \in \Pi \Rightarrow [A,B] \in \Pi$,

$\mathfrak{A} \cup [[x]] \in \mathfrak{z}$ and $\mathfrak{B} \cup [[x]] \in \mathfrak{z} \Rightarrow \mathfrak{A} \cup \mathfrak{B} \in \mathfrak{z}$.

A proximity (contiguity) induces a closure operator c_Π ($c_\mathfrak{z}$) on X as follows:

$$c_\Pi(A) = [x: [[x],A] \in \Pi] ,$$

$$c_\mathfrak{z}(A) = [x: [[x],A] \in \mathfrak{z}] .$$

Let (X,c) be a closure space and let c satisfy

$$x \in c([y]) \Leftrightarrow y \in c([x]) ,$$

a condition which is satisfied by all "induced" closure operators c_Π and $c_\mathfrak{z}$. The elementary proximity Π_c^1 on (X,c) is defined as follows

$$[A,B] \in \Pi_c^1 \Leftrightarrow c(A) \cap c(B) \neq \emptyset .$$

Analogously we define the elementary contiguity $\mathfrak{z}_c^1$ on (X,c) by

$$\mathfrak{A} \in \mathfrak{z}_c^1 \Leftrightarrow |\mathfrak{A}| < \aleph_0 \text{ and } \cap[c(A): A \in \mathfrak{A}] \neq \emptyset .$$

A grill $\mathfrak{G}$ which is such that for all $[A,B] \subset \mathfrak{G}$ $[A,B] \in \Pi$ is called a Π - clan and a grill $\mathfrak{G}'$ for which all finite subfamilies $\mathfrak{A}$ of $\mathfrak{G}'$ satisfy $\mathfrak{A} \in \mathfrak{z}$ is called a $\mathfrak{z}$ - clan.

Whether a proximity (contiguity) is an RI - proximity (contiguity) can be described in terms of properties of adherence grills.

Lemma 3. A proximity Π on X is an RI - proximity iff for all $x \in X$ $\mathfrak{G}_{c_\Pi}(x)$ is a Π - clan. A contiguity $\mathfrak{z}$ on X is an RI - contiguity iff for all $x \in X$ $\mathfrak{G}_{c_\mathfrak{z}}(x)$ is a $\mathfrak{z}$ - clan.

We also note the following characterization for separated RI - structures.

Lemma 4. An RI - proximity (contiguity) is separated iff (X,c_Π) (or $(X,c_\mathfrak{z})$) is a G_0 - space.

It is known [6] that if $[A,B] \in \Pi$ then there exists a maximal Π - clan containing $[A,B]$ and that if Π is an RI - proximity then all $\mathfrak{G}_{c_\Pi}(x)$ are maximal Π - clans. Similarly [3], $\mathfrak{A} \in \mathfrak{z}$ implies the existence of a maximal $\mathfrak{z}$ - clan containing $\mathfrak{A}$. Using Lemma 4 one easily shows that the $\mathfrak{G}_{c_\mathfrak{z}}(x)$ are maximal $\mathfrak{z}$ - clans if $\mathfrak{z}$ is an RI - contiguity.

Now let X^Π be the family of all maximal Π - clans with respect to a given RI - proximity Π . Then

$$[\mathfrak{G}_{c_\Pi}(x) : x \in X] \subset X^\Pi .$$

Further let

$$A^\Pi = [\mathfrak{G} : A \in \mathfrak{G} \in X^\Pi]$$

and define

$$h^\Pi(\alpha) = (\varphi(\alpha))^\Pi \cup \alpha , \quad \text{for all} \quad \alpha \subset X^\Pi ,$$

then $(\varphi , (X^\Pi , h^\Pi))$ is a G_0 - extension of (X, c_Π) if Π is a separated RI - proximity. Moreover, $[A,B] \in \Pi$ iff there exists a $\mathfrak{G} \in X^\Pi$ with $[A,B] \in \mathfrak{G}$, iff $\mathfrak{G} \in h^\Pi(\varphi(A)) \cap h^\Pi(\varphi(B))$. We thus have proved:

Theorem 2. Let Π be a separated RI - proximity on X . Then there exists an extension $(\varphi , (X^\Pi , h^\Pi))$ of (X, c_Π) such that

$$[A,B] \in \Pi \quad \text{iff} \quad [\varphi(A), \varphi(B)] \in \Pi^1_{h^\Pi} .$$

In a completely analogous manner one establishes:

Theorem 3. Let ξ be a separated RI - contiguity on X . Then there exists an extension $(\varphi , (X^\xi , h^\xi))$ of (X, c_ξ) such that

$$\mathfrak{A} \in \xi \quad \text{iff} \quad [\varphi(A) : A \in \mathfrak{A}] \in \xi^1_{h^\xi} .$$

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