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RELATIVE CONNECTEDNESSES AND DISCONNECTEDNESSES
IN TOPOLOGICAL CATEGORIES

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Starting with works of the author, A.V. ARHANGEL'SKII and R. WIEGANDT [1] have studied connectednesses and disconnectednesses in topology. They have stated necessary and sufficient conditions for the case that a class of topological spaces is a connectedness or disconnectedness respectively. Correspondingly, the author characterizes relative connectednesses and disconnectednesses in topological categories. Therefore the results may be applied to nearness spaces which have been developped by H. HERRLICH [3]. The relativization of connectedness is obtained in a natural way if one looks for a concept of connectedness such that the corresponding components may be identified with the quasicomponents introduced by F. HAUSDORFF [2].

Let $\underline{C}$ be a concrete topological category in the sense of HERRLICH [3], which is properly fibered. Thus initial and final structures exist and they are uniquely determined by their defining properties.

Let $\underline{C}_{(2)}$ be the <u>category of pairs</u> with respect to $\underline{C}$, i.e.

(1) objects of $\underline{C}_{(2)}$ are pairs $((X,\xi),(Y,\eta))$ where $(X,\xi)$ is an object in $\underline{C}$, $Y$ a subset of $X$ and $\eta$ the initial structure with respect to $(Y,i,(X,\xi))$ where $i\colon Y \to X$ is the inclusion map.

(2) morphisms $f\colon ((X,\xi),(Y,\eta)) \to ((X',\xi'),(Y',\eta'))$ are morphisms $f\colon (X,\xi) \to (X',\xi')$ in $\underline{C}$ such that $f[Y] \subset Y'$.

For each subclass $\underline{E}$ of $|\underline{C}|$ let us define
$C_{rel}\ \underline{E} = \{(X,Y) \in |\underline{C}_{(2)}| : Y$ is $\underline{E}$-connected with respect to $X$, i.e. $f|Y$ is constant for each $E \in \underline{E}$ and each $\underline{C}$-morphism $f\colon X \to E\}$.

$\underline{K} \subset |\underline{C}_{(2)}|$ is called a <u>relative connectedness</u>, iff $\underline{K} = C_{rel}\ \underline{E}$ for some $\underline{E} \subset |\underline{C}|$.

For each subclass $\underline{K}$ of $|C_{(2)}|$ let us define

$D_{rel}\ \underline{K} = \{\ Z \in |\underline{C}| : f|Y$ is constant for each $\underline{C}$-morphism $f: X \to Z$ and each $Y \subset X$ satisfying $(X,Y) \in \underline{K}\ \}$.

$E \subset |\underline{C}|$ is called a <u>relative disconnectedness</u>, iff $\underline{E} = D_{rel}\ \underline{K}$ for some $\underline{K} \subset |C_{(2)}|$.

The following two theorems characterize relative connectednesses and disconnectednesses.

<u>Theorem 1:</u> Let $\underline{K}$ be a subclass of $|\underline{C}_{(2)}|$. Then the following are equivalent:

(1) $\underline{K}$ is a relative connectedness.

(2) $\underline{K} = \underline{PK} := C_{rel}\ D_{rel}\ \underline{K}$.

(3)(a) $\{(X,Y) \in |\underline{C}_{(2)}| :$ Y consists at most of a single element $\} \subset \underline{K}$

(b) Let $(X,Y) \in \underline{K}$, and $f: (X,Y) \to (X',Y')$ a $\underline{C}_{(2)}$-morphism such that $Y' = f[Y]$. Then $(X',Y') \in \underline{K}$.

(c) Let $(X,A_i) \in \underline{K}$ for each i belonging to an index set I, and $\bigcap_{i \in I} A_i \neq \emptyset$. Then $(X, \bigcup_{i \in I} A_i) \in \underline{K}$.

(d) Let $f: (X,Y) \to (X',Y')$ be a quotient map *) such that $f[Y] = Y'$. Further let $(X',Y') \in \underline{K}$, and $(X, f^{-1}(x')) \in \underline{K}$ for each $x' \in X'$. Then $(X,Y) \in \underline{K}$.

*) that means $f: X \to X'$ is surjective and the $\underline{C}$-structure on X' coincides with the final structure with respect to f.

Theorem 2: Let $\underline{E}$ be a (isomorphism-closed) subclass of $|\underline{C}|$. Then the following are equivalent:

(1) $\underline{E}$ is a relative disconnectedness.

(2) $\underline{E} = Q\underline{E} := D_{rel}\, C_{rel}\, \underline{E}$.

(3) $\underline{E}$ is closed under formation of weak subobjects**) and products.

Remarks: 1 Let us look at the first theorem:

a) The operator P is a hull operator, i.e. P is extensive, isotone and idempotent.

b) Let $\underline{K}$ be a subclass of $|\underline{C}_{(2)}|$ satisfying (3)(a) and (c). Then each $\underline{C}$-object may be decomposed into maximal subsets M such that $(X,M) \in \underline{K}$, the so-called $\underline{K}$-quasicomponents. Let $\underline{C}$ be the category Top of topological spaces and continuous maps and let $\underline{K} = C_{rel}\,\{D_2\}$, where $D_2$ is the two-point discrete space. Then the $\underline{K}$-quasicomponents may be identified with the quasicomponents introduced by HAUSDORFF.

2 Let us look at the second theorem: Evidently, $Q\underline{E}$ is the extremal epireflective hull of $\underline{E}$. The hull operator Q is obtained as a composition of the two operators $C_{rel}$ and $D_{rel}$ which establish a Galois-correspondence between the relative connectednesses of $|C_{(2)}|$ and the relative disconnectednesses of $|\underline{C}|$ ( i.e. a 1-1-correspondence which converts the inclusion relation ).

**) in the sense of [4].

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