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REMARKS TO THE PROBLEM OF DEFINING A TOPOLOGY BY ITS HOMEOMORPHISM GROUP

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1. Notation. $\emptyset$ is the empty set. For any set X we denote by $|X|$ the cardinality of X, by $\mathcal{P}(X) := \{\ldots, M, \ldots\}$ the powerset of X and by $\mathcal{S}(X) := \mathcal{P}[\mathcal{P}(X)] := \{\ldots, \mathcal{T}, \ldots\}$. Let $\mathrm{bij}(X,Y) := \{f : X \to Y \mid f \text{ is bijective}\}$. Let $\mathrm{bij}(X) := \mathrm{bij}(X,X)$ be the group of all bijections of X onto itself and $\mathcal{G}(X)$ the set of all subgroups of bij(X). Finally $\mathcal{G}$ resp. $\mathcal{T}$ denotes the categories of all groups resp. topological spaces.

2. Introduction. At the beginning we want to refer to problems which frequently arise but are infrequently recognized. To gain adequate generality we use concrete categories ([7], p. 13-14). A concrete category is a triple $\mathcal{C} := (\mathcal{O}, F, \mathrm{mor})$. $\mathcal{O}$ are the $\mathcal{C}$-objects, $F : \mathcal{O} \to \mathcal{M}$ is a set-valued function ($\mathcal{M}$ is the class of all sets and for each $\mathcal{C}$-object Ξ, $F(\Xi)$ is called the carrier or underlying set of Ξ), and $\mathrm{mor} : \mathcal{O} \times \mathcal{O} \to \mathcal{M}$ is a set-valued function, where for each pair (Ξ, Ω) of $\mathcal{C}$-objects, $\mathrm{mor}(\Xi, \Omega) \subseteq F(\Omega)^{F(\Xi)}$ is called the set of $\mathcal{C}$-morphisms with domain Ξ and codomain Ω which satisfies the usual conditions. All subsequent categories $\mathcal{C}$, $\mathcal{K}$, ... are concrete and we denote the objects of $\mathcal{C}$, for shortness, by (X, Ξ) with $X := F(\Xi)$. Finally we notice, that to every (X, Ξ) we can associate the group $\mathrm{aut}(X, \Xi) \in \mathcal{G}(X)$, the automorphism- or symmetry group of (X, Ξ). This group is of fundamental importance for a deeper insight into the nature of (X, Ξ) (see for example [16], p.142). This motivatives immediately the following general representation problem.

(D) Given a category $\mathcal{C}$ and a group $G \in \mathcal{G}$. Does there exist

$$\text{a } (X, \Xi) \in \mathcal{C} \text{ such that } \mathrm{aut}(X, \Xi) \simeq G \ ?$$

It seems that this question was raised in a still more general setting for the first time in 1955 by M.Gerstenhaber [3]. Hitherto the problem (D) could be answered affirmatively for some categories, particularly in the important case $\mathcal{T}$ (J.De Groot [5], Z.Hedrlín - E.Mendelson [6] and M.C.Thornton [14]).

Because of the special structure of the automorphism groups we can further use them to compare and connect objects of very different categories. For example, let $\mathcal{C}$ and $\mathcal{K}$ be categories, $(X,\Xi)\in\mathcal{C}$ and $(X,\Omega)\in\mathcal{K}$ be objects with the same carrier. We call Ξ and Ω <u>compatible on X</u>, iff

$$\mathrm{aut}(X,\Xi) = \mathrm{aut}(X,\Omega) .$$

This concept creates naturally the following "inverse problem": Given a set X and $(X,\Omega)\in\mathcal{K}$. Find a compatible $(X,\Xi)\in\mathcal{C}$.

This question is a special case of a problem which is an essential refinement of (D), the <u>realisation problem</u> (on X):

(<u>R</u>). Given a category $\mathcal{C}$, a set X and a $G\in\mathcal{G}(X)$. Does there exist a $(X,\Xi)\in\mathcal{C}$ such that

$$\mathrm{aut}(X,\Xi) = G \ ?$$

<u>Remarks</u>. <u>1</u>) We want to call attention to two facts in (R). First, G is always a concrete group of bijections of X onto itself and second, we demand the equality of both $\mathrm{aut}(X,\Xi)$ and G and not just the abstract isomorphism.

<u>2</u>) By the <u>topological realisation problem (T)</u> we understand the problem (R) for $\mathcal{C} := \mathcal{T}$. A strong impulse to look at the problem (T) came from the theory of relativity. Since some prominent mathematicians and physicists criticized the use of locally euclidean topologies in mathematical models of space-time, the way to introduce a topology into a space with an "indefinite metric" must be thought over again. A milestone to this reflection is the important paper of E.C.Zeeman [18] in 1967 which since then stimulated and influenced all results on this subject. In [18] a new topology $\mathcal{Z}$, with various very remarkable and attractive matematical and physical properties is introduced into the space-time of the special theory cf relativity instead of the usual euclidean topology. We mention just one interesting fact: The full homeomorphism group $H(R^4,\mathcal{Z})$ is identical with the causality group, which is generated by the POINCARÉ-group and the dilatations. Thus the causality structure and ZEEMAN'S topology $\mathcal{Z}$ are compatible on their carrier.

3) The problem (T) can be considered as a question in the sense of F.KLEIN´S "Erlanger Programm". The roots of (T) can be traced back to the "problem of N.WIENER" [17] (Given an infinite set X and a preassigned $G \in \mathcal{S}(X)$. Construct all topologies $\mathcal{X}$ on X with $G \subseteq H(X,\mathcal{X})$. [1] , [8]) and to the "modification problem" which grew out of discussion between C.J.Everett, J.v.Neumann, E.Teller and S.M.Ulam [2] (Given $(X,\mathcal{X}) \in \mathcal{T}$. Does there exist a topology $\mathcal{Y} \neq \mathcal{X}$ on X, such that $H(X,\mathcal{X}) = H(X,\mathcal{Y})$? [9] - [11], [12] , [15] , p.32).

3. The realisation problem for the category $\mathcal{S}$. We start by recalling the definition of the concrete category $\mathcal{S}$ ([13]). The objects of $\mathcal{S}$ are pairs $(X,\mathcal{X})$, where X, the carrier of $(X,\mathcal{X})$, is a set and $\mathcal{X} \subseteq \mathcal{S}(X)$ is a set of subsets of X. If $(X,\mathcal{X})$, $(Y,\mathcal{Y}) \in \mathcal{S}$ then a map $f : X \to Y$ is called a morphism ,

$f \in \text{mor}[(X,\mathcal{X}), (Y,\mathcal{Y})]$, if

$$\forall Q \in \mathcal{Y} \ : \ f^{-1}(Q) \in \mathcal{X}$$

resp. an isomorphism, $f \in \text{iso}[(X,\mathcal{X}), (Y,\mathcal{Y})]$, if f is bijection, $f \in \text{mor}[(X,\mathcal{X}), (Y,\mathcal{Y})]$, and $f^{-1} \in \text{mor}[(Y,\mathcal{Y}), (X,\mathcal{X})]$. $(X,\mathcal{X})$ and $(Y,\mathcal{Y})$ are called isomorphic, $(X,\mathcal{X}) \sim (Y,\mathcal{Y})$, if $\text{iso}[(X,\mathcal{X}),(Y,\mathcal{Y})] \neq \emptyset$. For every $(X,\mathcal{X}) \in \mathcal{S}$ we denote by

$$H(X,\mathcal{X}) := \text{iso}[(X,\mathcal{X}), (X,\mathcal{X})] \tag{1}$$

its full automorphism group. Finally we want to point out the following useful convention. Let X and Y be sets. Then every $f \in \text{bij}(X,Y)$ induces a canonical bijection

$$f^{\#} : \mathcal{S}(X) \to \mathcal{S}(Y) \ ; \qquad f^{\#}(Q) := \{ f(x) \mid x \in Q \} . \tag{2}$$

Therefore we can characterize those $f \in \text{bij}(X)$ which are in $H(X,\mathcal{X})$ by the equivalence

$$f \in H(X,\mathcal{X}) \iff f^{\#}(\mathcal{X}) = \mathcal{X} . \tag{3}$$

The realisation problem in $\mathcal{S}$ (here we take in account just the case $\mathcal{S} = \mathcal{S}'$) then reads

(S). Let X be a set and $G \in \mathcal{S}(X)$. Does there exist a

$$\mathcal{X} \subseteq \mathcal{S}(X) \ \text{ such that}$$

$$H(X,\mathcal{X}) = G\ ?$$

Remarks. 1) $\mathcal{T}$ is a full subcategory of $\mathcal{S}$.
2) In all subsequent considerations we concern ourself exclusively with the study of (S) which is in some sense a generalisation of (T). The main reason for doing this is the greater simplicity.
3) In general (S) need not to possess any solution. Put, for example, $X := \{1,2,3\}$ and $G := \{(1), (1,2,3), (1,3,2)\} \in \mathcal{S}(X)$. It is easy to show that there does not exist any $\mathcal{X} \subseteq \wp(X)$ such that $H(X,\mathcal{X}) = G$.
4) Easy examples show that the existence does not imply uniqueness.

4. A lemma on posets. In this section we formulate and prove a rather technical lemma. Let $(Y,\leq)$ be a poset (i.e. $\leq$ is a reflexive and transitive relation on Y),

$$\Pi := \mathrm{aut}(Y,\leq)$$

its full automorphism group, $\mathcal{Y} := \{\ldots,\{y\},\ldots\}$ the set of all atoms respectively $\mathcal{O}\!\mathcal{l} := \{\ldots, \bar{Y} := Y - \{y\}, \ldots\}$ the set of all antiatoms in the usual boolean lattice $(\wp(Y),\subseteq)$, and

$$\Sigma := \mathrm{aut}\,[\wp(Y),\subseteq].$$

Further we consider mappings

$$J : \mathrm{bij}(Y) \to \Sigma\ ; \quad J(g) := g^{\#}$$

(J is a group-isomorphism) and

$$j : Y \to \mathcal{Y}\ ; \quad j(y) := \{y\}$$

(j is a bijection) and denote by

$$A := J(\Pi)$$

whereby A is a subgroup of Σ. Trivially, we can use j to transfer the partial order $\leq$ from Y to $\mathcal{Y}$: For $U,V \in \mathcal{Y}$ we define

$$U \leq V\ :\Longleftrightarrow\ j^{-1}(U) \leq j^{-1}(V)\ ;$$

then $(Y,\leq)$ and $(\mathcal{Y},\leq)$ are order-isomorphic. A direct computation then delivers a simple characterisation of A in Σ.

Lemma 1. Let $f \in \Sigma$. Then

$$f \in \mathfrak{A} \Longleftrightarrow f|\mathcal{Y} \in \mathrm{aut}(\mathcal{Y}, \leq) .$$

Now we prove

Lemma 2. For $U, V \in \mathcal{G}(Y)$ we define the relation $U \prec V$ to be true, iff

$$U = \emptyset \ ; \text{ or}$$

$$V = Y \ ; \text{ or}$$

$$U \leq V, \text{ if } U \in \mathcal{Y} \text{ and } V \in \mathcal{Y} \ ; \text{ or}$$

$$\exists\, W \in \mathcal{Y} : U \leq W \subseteq V , \text{ if } U \in \mathcal{Y} \text{ and } V \notin \mathcal{Y} ; \text{ or}$$

$$U \subseteq V , \text{ if } U \notin \mathcal{Y} \text{ and } V \notin \mathcal{Y} .$$

The the following statements are valid:

1) $\prec$ is a partial order in $\mathcal{G}(Y)$ which is finer than $\subseteq$.

2) $\prec$ is antisymmetric iff $\leq$ is antisymmetric.

3) $\mathfrak{A}$ is also the set of all antiatoms in $(\mathcal{G}(Y), \prec)$.

4) If $(Y, \leq)$ is a join-semilattice ([4], p.8) then also $(\mathcal{G}(Y), \prec)$.
Foer every $U, V \in \mathcal{G}(Y)$ with $U \not\prec V$ and $V \not\prec U$ we have

$$\sup_{\prec}(U,V) = \begin{cases} \sup_{\leq} (U,V), & \text{if } U \in \mathcal{Y} \text{ and } V \in \mathcal{Y} . \\ U \cup V , & \text{otherwise.} \end{cases}$$

5) $$\mathrm{aut}\, [\mathcal{G}(Y), \prec] = \mathfrak{A} .$$

Remark. In general $\inf_{\prec}(U,V)$ does not exist, even if $(Y, \leq)$ is a boolean lattice.

Proof. 1) - 3) . Straightforward.

4). The case $U \prec V$ is trivial. Now suppose $U \not\prec V$ and $V \not\prec U$.

α) $U \in \mathcal{Y}$ and $V \in \mathcal{Y}$. Because of the existence of $S := \sup_{\leq}(U,V) \in \mathcal{Y}$, S is an $\prec$-upper bound for U and V.

Assertion: $\exists\, T \in \mathcal{G}(Y) : U \prec T \prec S,\ V \prec T \prec S$.

Then $T \in \mathcal{Y}$ and therefore $T = S$ which implies

$$\exists \ \sup_{\prec}(U,V) = \sup_{\leq}(U,V)$$

β). $V \notin \mathcal{Y}$ (trivially, $V \neq \emptyset$). We put $S := U \cup V$; again, S is an $\prec$-upper bound for U and V.
Assertion: $\exists \ T \in \mathcal{Y}(Y)$: $U \prec T \prec S$, $V \prec T \prec S$.
In our case, the second term is equivalent to

$$V \subseteq T \subseteq U \cup V \ .$$

If $U \notin \mathcal{Y}$, then because of

$$U \subseteq T \subseteq U \cup V$$

we obtain $T = U \cup V = S$ and therefore

$$\exists \ \sup_{\prec}(U,V) = U \cup V \ .$$

If $U = \{u\} \in \mathcal{Y}$, then because of

$$V \subseteq T \subseteq V \cup \{u\}$$

and $U \prec T$ we obrain $T = V \cup \{u\} = S$ and therefore again

$$\exists \ \sup_{\prec}(U,V) = U \cup V \ .$$

5). First we show

(1) $$A^* := \operatorname{aut} [\mathcal{Y}(Y), \prec] \subseteq \Sigma.$$

To do this we must carry out several steps.

(2) $$\forall f \in A^* \ \forall \ Q \in \mathcal{O} \ : f(Q) \in \mathcal{O}$$

Assume the existence of a $Q \in \mathcal{O}$ such that $f(Q) \notin \mathcal{O}$; because $f(Q) \neq Y$ there must exist a $T \in \mathcal{Y}(Y)$ with

$$f(Q) \underset{\neq}{\prec} T \underset{\neq}{\prec} Y \ .$$

Together with $f \in A^*$ we also have $f^{-1} \in A^*$ and therefore

$$\exists f^{-1}(T) \in \mathcal{Y}(Y) : \ Q = f^{-1}f(Q) \underset{\neq}{\prec} f^{-1}(T) \underset{\neq}{\prec} f^{-1}(Y) = Y$$

so that $Q \notin \mathcal{O}$. This is a contradiction to $Q \in \mathcal{O}$. This proves (2).
An immediate consequence of (2) is

(3) $$\forall f \in A^* \ : \ f(\mathcal{O}) = \mathcal{O} \ .$$

Now we claim

(4) $$\forall f \in A^* \ \forall \ Q \notin \mathcal{Y} \ : f(Q) \notin \mathcal{Y} \ .$$

Assume the existence of a $f \in A^*$ and a $Q \notin \mathcal{Y}$ such that $f(Q) := U := \{u\} \in \mathcal{Y}$. Trivially, $Q \neq \emptyset$ and therefore $|Q| \geq 2$. We choose two elements $v, w \in Q$ with $v \neq w$ and put $V := \bar{v} \in \mathcal{U}$ and $W := \bar{w} \in \mathcal{U}$. Because of (2) there exist two elements $r, s \in Y$ with $r \neq s$ such that $f(V) = \bar{r} \in \mathcal{U}$ and $f(W) := \bar{s} \in \mathcal{U}$.

$$\sup_{\prec}(V,Q) = Y \Longrightarrow f[\sup_{\prec}(V,Q)] = \sup_{\prec}[f(V),f(Q)] = f(Y) =$$
$$= Y \Longrightarrow \sup_{\prec}(\bar{r},\{u\}) = \bar{r} \cup \{u\} = Y \Longrightarrow u \notin \bar{r} \Longrightarrow$$
$$u = r$$

In an analogous manner $\sup_{\prec}(W,Q) = Y$ implies

$$u = s .$$

This is a contradiction to $r \neq s$. Therefore (4) is correct. This result is equivalent to

(5) $$\forall f \in A^{*} : f(\mathcal{Y}) = \mathcal{Y} .$$

With (5) we can sharpen (2).

(6) $$\forall f \in A^{*} \quad \forall y \in Y : f(\bar{y}) = Y - f(\{y\}) .$$

Let $f \in A^{*}$ and $y \in Y$. Now (3) implies

$$\exists u \in Y : f(\bar{y}) = \bar{u} \in \mathcal{U}$$

and (5) implies

$$\exists v \in Y : f(\{y\}) = \{v\} \in \mathcal{Y} .$$

Therefore we obtain

$$\sup_{\prec}(\bar{y},\{y\}) = Y \Longrightarrow f[\sup_{\prec}(\bar{y},\{y\})] = \sup_{\prec}[f(\bar{y}), f(\{y\})] =$$
$$= f(Y) = Y \Longrightarrow \sup_{\prec}(\bar{u},\{v\}) = Y \Longrightarrow u = v \Longrightarrow$$
$$f(\bar{y}) = Y - f(\{y\}) .$$

This proves (6). Finally we show

(7) $$\forall f \in A^{*} : u \in U \Longrightarrow f(\{u\}) \subseteq f(U).$$

Let $f \in A^{*}$ and $u \in U$. Then with (5) we have

$$\exists v \in Y : \{v\} = f(\{u\}) \prec f(U) := V .$$

We claim

$$v \in V$$

Assume, on the contrary, that $v \notin V$. Then together with (6) and Lemma 2, 4) have

$$\sup_{\prec}(\bar{v},V) = \bar{v} \Longrightarrow f^{-1}[\sup_{\prec}(\bar{v},V)] = \sup_{\prec}[f^{-1}(\bar{v}), f^{-1}(V)] =$$
$$= f^{-1}(\bar{v}) \Longrightarrow \sup_{\prec}[Y - f^{-1}(\{v\}), f^{-1}(V)] = Y - f^{-1}(\{v\}) \Longrightarrow$$
$$(Y - \{u\}) \cup U = Y - \{u\} \Longrightarrow U \subseteq Y - \{u\} \Longrightarrow u \notin U .$$

This is a contradiction to $u \in U$. Therefore $f(\{u\}) \subseteq f(U)$.

After this preparation we prove (1). Let $U,V \in \mathcal{Y}(Y)$ with $U \subseteq V$ and $f \in \mathcal{A}^*$.

α) If $U \notin \mathcal{Y}$ and $V \notin \mathcal{Y}$, then together with (5) we get

$U \subseteq V \Longleftrightarrow U \prec V \Longleftrightarrow f(U) \prec f(V) \Longleftrightarrow f(U) \subseteq f(V)$

β) If $U \in \mathcal{Y}$, then with (7) we get

$$U \subseteq V \Longrightarrow f(U) \subseteq f(V) .$$

Because of $f^{-1} \in \mathcal{A}^*$ we also get

$$U \subset V \Longrightarrow f^{-1}(U) \subseteq f^{-1}(V)$$

and therefore have again

$$U \subseteq V \Longleftrightarrow f(U) \subseteq f(V) .$$

This proves (1). Next we note

(8) $$\mathcal{A}^* \subseteq \mathcal{A} ,$$

which is a direct consequence of (1), (5) and Lemma 1. Now the last step is

$$\mathcal{A}^* \subseteq \mathcal{A} .$$

Let $U,V \in \mathcal{Y}(Y)$ with $U \prec V$ and $f \in \mathcal{A}$.

α) If $U \notin \mathcal{Y}$ and $V \notin \mathcal{Y}$, then together with (5) we get

$U \prec V \Longleftrightarrow U \subseteq V \Longleftrightarrow f(U) \subseteq f(V) \Longleftrightarrow f(U) \prec f(V)$.

β) If $U \in \mathcal{Y}$ and $V \in \mathcal{Y}$, then by the definition

$$\exists\, g \in \Pi \ : f = g^{\#}$$

and therefore, with $U := \{u\}$ and $V := \{v\}$,

$U \prec V \Longleftrightarrow U \leq V \Longleftrightarrow u \leq v \Longleftrightarrow g(u) \leq g(v) \Longleftrightarrow g^{\#}(\{u\}) =$
$= f(U) \prec g^{\#}(\{v\}) = f(V)$.

γ) If $U \in \mathcal{Y}$ and $V \notin \mathcal{Y}$, then we have

$U \prec V \Longleftrightarrow \exists\, W \in \mathcal{Y} \ : \ U \leq W \subseteq V \Longrightarrow f(U) \leq f(W) \subseteq f(V) \Longrightarrow$
$f(U) \prec f(U)$.

In an analogous manner we find
$U \prec V \Longrightarrow f^{-1}(U) \prec f^{-1}(V)$.
Therefore

$$U \prec V \Longleftrightarrow f(U) \prec f(V) .$$

This proves Lemma 2.

We close this section with a sharper version of Lemma 2,4).

Lemma 3. Let $(\mathcal{Y}(Y), \prec)$ be as in Lemma 2. If $(Y, \leq)$ is a complete join-semilattice, then the same is true for $(\mathcal{Y}(Y), \prec)$ (i.e., for any family $\{M_\lambda\}$ of subsets of $\mathcal{Y}(Y)$ there exists $\sup_\prec (M_\lambda)$).

Proof. Let $\Lambda \neq \emptyset$ be any fixed index set and

$$\forall \lambda \in \Lambda : M_\lambda \in \mathcal{Y}(Y) .$$

We divide Λ into two disjoint subsets $\Lambda := \Delta \cup \Omega$ such that $\lambda \in \Delta$ iff $M_\lambda \in \mathcal{M} \cup \{\emptyset\}$ and $\lambda \in \Omega$ otherwise. We then distinguish three cases.

1) $\Delta = \emptyset \Longrightarrow \exists \sup_\prec M_\lambda = \bigcup_{\lambda \in \Lambda} M_\lambda$.

2) $\Omega = \emptyset \Longrightarrow \exists \sup_\prec M_\lambda = \sup_\leq M_\lambda$.

3) $\Delta \neq \emptyset$ and $\Omega \neq \emptyset$.

If we put for brevity

$\{u\} := \sup_{\leq, \delta \in \Delta} M_\delta$, $V := \bigcup_{\omega \in \Omega} M_\omega$ and $S := \sup_\prec (\{u\}, V)$, then

trivially, S is an upper $\prec$-bound for $\{M_\lambda\}$. If we assume

$$\exists T \in \mathcal{Y}(Y) \quad \forall \lambda \in \Lambda : M_\lambda \prec T \prec S$$

then a straightforward calculation shows that $T = S$. This proves Lemma 3.

5. The space $(\mathcal{S}(X), \prec)$. For the whole section let X be a fixed set and $G \in \mathcal{Y}(X)$. Again we consider $(\mathcal{Y}(X), \subseteq)$ and $(\mathcal{S}(X), \subseteq)$ as boolean lattices and denote their automorphism groups (with respect to their partial order) by $\Pi := \mathrm{aut}[\mathcal{Y}(X), \subseteq]$ and $\Sigma := \mathrm{aut}[\mathcal{S}(X), \subseteq]$. In addition we put

$$I : \mathrm{bij}(X) \to \Pi \; ; \; I(f) := f^{\#}$$
$$J : \mathrm{bij}[\mathcal{Y}(X)] \to \Sigma \; ; \; J(g) := g^{\#}$$
$$K : J \cdot I : \mathrm{bij}(X) \to \Sigma$$

(I and J are group isomorphisms and K is a group monomorphism) and

(1) $$A := K\,\mathrm{bij}(X) = J(\Pi)$$
$$\Gamma := K(G)$$

(Γ is a subgroup of $\mathcal{A}$ and $\mathcal{A}$ a subgroup of Σ). Clearly, for any $\varphi \in \Sigma$ we have

$$(2) \qquad \begin{aligned} &\varphi \in \mathcal{A} \Longleftrightarrow \\ &\exists\, f \in \mathrm{bij}(X) : \varphi = (f^{\#})^{\#} \Longleftrightarrow \\ &\exists\, g \in \Pi \;:\; \varphi = g^{\#} . \end{aligned}$$

With (2) we can reformulate (S): The existence of a $\mathcal{X} \in \mathcal{S}(X)$ such that $H(X,\mathcal{X}) = G$ is equivalent to the following two conditions

$$(3) \qquad \begin{aligned} &\text{a)} \quad \forall \gamma \in \Gamma : \gamma(\mathcal{X}) = \mathcal{X} \\ &\text{b)} \quad \forall \alpha \in \mathcal{A} - \Gamma : \alpha(\mathcal{X}) \neq \mathcal{X} . \end{aligned}$$

In this case we call $\mathcal{X}$ a <u>maximal fixed point of</u> Γ <u>with respect to</u> $\mathcal{A}$. If we consider the group $\mathcal{A}$ as an action on $\mathcal{S}(X)$, i.e. $(\mathcal{A}, \mathcal{S}(X))$ as a transformation group, because of (2) we finally obtain

<u>Theorem 1.</u> The following statements are equivalent

1. $\exists\, \mathcal{X} \in \mathcal{S}(X) : H(X,\mathcal{X}) = G$.
2. Γ is a stability subgroup of $(\mathcal{A}, \mathcal{S}(X))$.
3. There exist a maximal fixed point of Γ with respect to $\mathcal{A}$.

Now we are interested in an intrinsic characterization of $\mathcal{A}$. This can easily be done with Lemma 2. At first we identify $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{S}(X), \subseteq)$ with the posets $(Y, \leq)$ and $(\mathcal{P}(Y), \subseteq)$ respectively. Then we denote by

$$\mathfrak{A} := \{ \mathcal{T} \in \mathcal{S}(X) \mid \exists\, T \in \mathcal{P}(X) : \mathcal{T} = \{T\} \}$$

the atoms of $(\mathcal{S}(X), \subseteq)$. Finally, we define as in Lemma 2, the partial order $\prec$: For $\mathcal{U}, \mathcal{V} \in \mathcal{S}(X)$ we have $\mathcal{U} \prec \mathcal{V}$, iff

$$\begin{aligned} &\mathcal{U} = \emptyset , \text{ or} \\ &\mathcal{V} = \mathcal{P}(X) , \text{ or} \\ &U \leq V, \text{ if } \mathcal{U} := \{U\} \in \mathfrak{A} \text{ and } \mathcal{V} := \{V\} \in \mathfrak{A} , \text{ or} \\ &\exists\, W \in \mathcal{P}(X) : U \leq W \in \mathcal{V} , \text{ if } \mathcal{U} := \{U\} \in \mathfrak{A} \text{ and } \mathcal{V} \notin \mathfrak{A}, \text{ or} \\ &\mathcal{U} \subseteq \mathcal{V} \text{ if } \mathcal{U} \notin \mathfrak{A} \text{ and } \mathcal{V} \notin \mathfrak{A} . \end{aligned}$$

With (1) and the results of the last section we obtain immediately

Theorem 2. 1) $\mathcal{A} = \mathrm{aut}[\mathcal{T}(X), \prec]$.
2) $(\mathcal{T}(X), \prec)$ is a complete join-semilattice.

Summing up the results of Theorems 1 and 2 we can recognize the realisation problem (S) as a special case of the following problem (identify Z with $\mathcal{T}(X)$ and $<$ with $\prec$).

(P) Given a complete join-semilattice $(Z, <)$, its full automorphism group $\mathcal{A} := \mathrm{aut}(Z, <)$ and let Γ be a subgroup of $\mathcal{A}$. Does Γ appear as a stability subgroup in the transformation group $(\mathcal{A}, Z)$? Or, in other words, does there exist a maximal fixed point of Γ with respect to $\mathcal{A}$ (see (3) a),b) ?

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