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Let us regard three disjoint arcs in the plane. Then it has no sense to speak about the concept of „between" among them. But in certain families of disjoint plane arcs the situation is quite different.

Let us take the system Σ^1 of segments

$S_n^1 = \{(x,y)\ ;\ x = \frac{1}{n}\ ,\ 0 \leqq y \leqq 1\}$ and

$S_0^1 = \{(x,y)\ ;\ x = 0\ ,\ 0 \leqq y \leqq 1\}$

and the system Σ^2 of segments

$S_n^2 = \{(x,y)\ ;\ x = (-1)^n \cdot \frac{1}{n}\ ,\ 0 \leqq y \leqq 1\}$ and

$S_0^2 = S_0^1$.

We can say that S_{n+1}^1 lies between S_n^1 and S_0^1 and S_0^2 lies between S_n^2 and S_{n+1}^2 for sufficiently large n- s. This visibly clear fact can be expressed exactly in the following way:

Let Σ be a system of disjoint plane arcs. The pair (U,V) of nonempty connected open sets of the plane is called <u>regular</u>, if $V \subset U$ and if for any $S \in \Sigma$ $S \cap V$ is contained in a component of $S \cap U$. Let (U,V) be a regular pair and S_1 , S_2 , S_3 three distinct members of Σ intersecting V . We say that S_2 lies between S_1 and S_3 mod (U,V) if $S_1 \cap V$ and $S_3 \cap V$ are contained in different components of $U \setminus S_2$. And now taking the systems Σ^i it is true that for any cut point q of S_0^i there exists an $\varepsilon > 0$ such that for each regular pair (U,V) satisfying the condition $q \in V$ and $\operatorname{diam} U < \varepsilon$ from the fact that S_n^i and S_{n+1}^i intersect V follows in the case $i = 1$ that S_{n+1}^1 lies between S_n^1 and S_0^1 mod (U,V) and in the case $i = 2$ that S_0^2 lies between S_n^2 and S_{n+1}^2 mod (U,V) . These properties hold if we replace the systems Σ^i by $\varphi(\Sigma^i)$ where φ is an arbitrary autohomeomorphism of the plane.

The base space in both systems Σ^i is the set of integers with the limit point 0 . Setting k between j and m mod (U,V)

if S^i_k lies between S^i_j and S^i_m mod (U,V) we get a system of triadic relations on the base space. If the regular pairs (U,V) are sufficiently small then this system of relations satisfies some useful conditions. To describe them first of all we give the definition of the ordered set.

Let M be a nonempty set and R a triadic relation on M. The pair (M, R) is called an <u>ordered set</u> if

a) $(a,b,c) \in R \Longrightarrow a, b, c$ are distinct elements of M.

b) $(a,b,c) \in R \Longrightarrow (c,b,a) \in R$.

c) For any three distinct elements a, b, c of M from the assertions $(a,b,c) \in R$, $(b,c,a) \in R$ and $(c,a,b) \in R$ one and only one is valid.

d) For any four distinct elements a, b, c, d of M $(a,b,c) \in R$ and $(a,b,d) \notin R$ implies $(c,b,d) \in R$.

Let (M, R) be an ordered set and $\emptyset \neq M' \subset M$. Then the restriction $(M, R)|_{M'}$ is defined as usual.

Let (M, R) be an ordered set and $p \in M$. Then $M \setminus \{p\}$ decomposes uniquely in two sets M_1 and M_2 such that for any $b, c \in M \setminus \{p\}$ b and c are contained in different M_i - s iff $(b,p,c) \in R$. We say that M_1 and M_2 are the <u>sides</u> of p in (M, R).

And now giving the conditions mentioned above we obtain the concept of the locally ordering.

Let X be a topological space. The family $\Theta = \{(M_\alpha, R_\alpha ; \alpha \in A\}$ of ordered sets is said to be a <u>locally ordering</u> of X if it satisfies the conditions:

i) The system $\{M_\alpha ; \alpha \in A\}$ is an open base for X.

ii) For any $q \in M$ and $(M_{\alpha'}, R_{\alpha'}), (M_{\alpha''}, R_{\alpha''}) \in \Theta$ with $q \in M_{\alpha'} \cap M_{\alpha''}$ there is an (M_α, R_α) in Θ such that $q \in M_\alpha \subset M_{\alpha'} \cap M_{\alpha''}$ and $(M_{\alpha'}, R_{\alpha'})|_{M_\alpha} = (M_{\alpha''}, R_{\alpha''})|_{M_\alpha} = (M_\alpha, R_\alpha)$.

iii) For any $(M_\alpha, R_\alpha) \in \Theta$ and $p \in M_\alpha$ the sides of p in (M_α, R_α) are open sets in X.

Two locally orderings Θ and Θ' of X are said to be <u>equivalent</u> - $\Theta \sim \Theta'$ - if $\Theta \cup \Theta'$ is also a locally ordering of X. The classes of equivalences of the locally orderings of X are called <u>locally ordered structures</u>. The topological space X together with a locally ordered structure on it is called a <u>locally ordered space</u>.

Let $f : X \to X$ be an autohomeomorphism of the topological space X and Θ a locally ordering of X. We say that f disturbs Θ if $f(\Theta) \neq \Theta$.

And now we shall describe some autohomeomorphisms of the Cantor discontinuum, which disturb each locally ordering on it.

First of all let X be a topological space and $g : X \to X$ an autohomeomorphism of X. A finite sequence $Z = (L_1, \dots, L_n)$ of nonempty disjoint closed subsets of X is said to be a g - cycle if Z is a covering of X and $g(L_i) = L_{i+1}$ for $i = 1, \dots, n-1$ and $g(L_n) = L_1$. The autohomeomorphism $g : X \to X$ is an absolutely cyclic map if for any open covering Ω of X there is a g - cycle Z which is a refinement of Ω.

And now we have the following theorems:

Theorem 1. The only infinite T_1 - space which possesses an absolutely cyclic map is the Cantor discontinuum.

Theorem 2. Each absolutely cyclic map of the Cantor discontinuum C disturbs every locally ordering on C.

A simple example for an absolutely cyclic map is the following: The points of C should be regarded as infinite sequences $(x_1, \dots, x_k, \dots)$ of the symbols 0 and 1. The map $f : C \to C$ should be defined by the formulas

$$f((0, x_2, \dots, x_k, \dots)) = (1, x_2, \dots, x_k, \dots)$$

$$f((\underset{1}{1}, \underset{2}{1}, \dots, \underset{s}{1}, 0, x_{s+2}, \dots)) = (0, 0, \dots, 0, 1, x_{s+2}, \dots)$$

$$f((1, 1, \dots, 1, \dots)) = (0, 0, \dots, 0, \dots).$$

f is absolutely cyclic.