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Set transformations with centralizers formed by closed deformations of quasi-discrete topological spaces

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SET TRANSFORMATIONS WITH CENTRALIZERS FORMED BY CLOSED DEFORMATIONS OF QUASI-DISCRETE TOPOLOGICAL SPACES

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1. In paper [7] the authors solved the problem of realizibility of a transformation monoid, on a set X, which is a centralizer (in the full transformation monoid $(X^X, .)$) of a selfmap of X by means of endomorphisms of unary algebras (with more unary operations). Here an analogous problem is treated for continuous closed selfmaps of quasi-discrete topological spaces. The mentioned problem can be considered as a special case of the problem of representability of semigroups (especially groups) by various kinds of morphisms of algebraic or topological structures. Problems of this type were solved by many authors (a list of results and other details can be found in [14]). The problem studied here is the following: To find necessary and sufficient conditions on a transformation f of a set A for the existence of a quasi-discrete topology τ on the set A such that the centralizer of f in the full transformation monoid of A coincides with the monoid of all continuous closed selfmaps of the space (A,τ).

For the description of a set transformation (i.e. a mapping of a set into itself) we use terms and notation from papers [7], [8], [9], [11], [12]. Thus by a unary algebra we mean a unary algebra with one unary operation (i.e. a pair (A,f), where $A \neq \phi$ and $f \in \in A^A$). Let n be a natural number. The n-th iteration of the mapping f is denoted by f^n. A unary algebra (B,g) is a subalgebra of (A,f) if $B \subseteq A$, $f(B) \subseteq B$ and $g = f/_B$. The algebra (A,f) is said to be connected if to every pair of elements $a,b \in A$ there exists a pair of non-negative integers m,n such that $f^m(a) = f^n(b)$; (A,f) is called disconnected in the opposite case. A maximal connected subalgebra of (A,f) is called a component of (A,f) and $(A,f) = \sum_{\iota\in J}(A_\iota,f_\iota)$ means that $\{(A_\iota,f_\iota) : \iota \in J\}$ is the system of all components of the algebra (A,f). The finite cardinal number card $\cap \{B \subseteq A : f_\iota(B) \subseteq B\} = R(A_\iota,f_\iota)$ is called the rank of the component (A_ι,f_ι), the set $\cap\{B \subseteq A : f_\iota(B) \subseteq B\}$ is called a cycle of the component (A_ι,f_ι) and it is denoted by $A_{f_\iota}^{\infty_2}$. Hence $R(A_\iota,f_\iota) = \operatorname{card} A_\iota^{\infty_2}$. Further, for a unary algebra (A,f) we put $A_f^0 = \{a \in A : f^{-1}(a) = \phi\}$ and by $A_{f_\iota}^{\infty_1}$ (or briefly $A_\iota^{\infty_1}$) we denote a set of elements of the component (A_ι,f_ι) with this property: $a \in A_\iota^{\infty_1}$ if there exists a sequence $\{x_i\}_{0<i<\omega_0}$, $x_i \in A_\iota$ such that $a = x_0$, $f(x_{i+1}) = x_i$ and $x_i \neq x_{i+1}$ for all i. The set $A_\iota^{\infty_1} \cup A_\iota^{\infty_2}$ is also called the kernel of (A_ι, f_ι) (cf. [7]). We denote the centralizer of the set transformation $f \in A^A$ by $C(f)$, i.e. $C(f)$ is the monoid of all endomorphisms of the algebra (A,f). If $(A,f) = \sum_{\iota \in J} (A_\iota, f_\iota)$ is a unary algebra with $R(A_\iota, f_\iota) \leqslant 1$ for all $\iota \in J$, we define a binary relation $\leqslant_f$ on A putting $a \leqslant_f b$ for $a,b \in A$ if there exists a non-negative integer n such that $f^n(a) = b$. Evidently, $(A, \leqslant_f)$ is a partially ordered set.

By a topological space (A,τ) we mean a topological space in the sense of [3], i.e. τ is a closure operation on A satisfying four usual axioms (cl 1 – cl 3, p.237 and cl 4, p. 250 in [3]). A continuous selfmap of a space (A,τ) is called briefly a deformation of (A,τ); the monoid of all closed deformations of (A,τ) (i.e. the monoid of all closed continuous mappings of the space (A,τ) into itself) is denoted by $S(A,\tau)$. A topological space (A,τ) is said to be quasi-discrete if τ coincides with its quasi-discrete modification, i.e. $\tau X = \bigcup_{x \in X} \tau\{x\}$ for every non-empty set $X \subseteq A$. More results concerning quasi-discrete spaces (called also saturated) can be found in [3] chap. V and [10].

2. Let A be a set, f a transformation of A. There are many quasi-discrete topologies on the set A which can be simply defined by means of the mapping f. A certain naturaly induced quasi-discrete topology is given by the closure system of subalgebras of (A,f) and another is the dual topology to the former one.These toplogies (called upper and lower respectively) are introduced and studied for the so called Pawlak's machines (partial unary algebras with one unary operation) in paper [2], where there are described some of their properties with respect to special properties of Pawlak's machines. In case when $(A,f) = \Sigma (A_\iota, f_\iota)$ with $R(A_\iota, f_\iota) \leqslant 1$ for each $\iota \in J$, the above mentioned topologies coincide respectively with the left and right topologies on the ordered set $(A, \leqslant_f)$, cf. [13]. It is easy to see that for every non-constant connected transformation f of the set A with $R(A,f) \leqslant 1$ it holds $C(f) \neq S(A,\tau)$ if τ is the upper or the lower topology on $(A, \leqslant_f)$.

Define for our purposes some other topologies on (A,f). Let (A,f) be an arbitrary unary algebra. Let n be a non-negative integer. Denote by $\tau_f^{(n)}$ a topology on A such that the least neighbourhood of a point $a \in A$ is the set $\{x \in A : f^k(x) = a$, where $k = n+1, n+2, \cdots\}$, i.e. if $\tau_f^{(n)}$ means a closure operation on A (in the sense of [3]) we have for each subset X of A

$$\tau_f^{(n)} X = X \cup \bigcup_{n+1 \leqslant k < \omega_0} f^k(X) .$$

Clearly, $\tau_f^{(0)}$ is the upper topology on (A,f). For the sake of brevity we write τ_f instead of $\tau_f^{(1)}$.

2.1. Proposition. *Let $(A,f) = \sum_{\iota \in J} (A_\iota, f_\iota)$ be a unary algebra, n a non-negative integer. $(A,\tau_f^{(n)})$ is a quasi-discrete topological space which is a T_0-space (i.e. a discrete space of Alexandroff) iff $R(A_\iota,f_\iota) \leqslant 1$ for each $\iota \in J$. The space $(A,\tau_f^{(n)})$ is connected iff the algebra (A,f) is connected and it is compact iff J is a finite set and each component of (A,f) has a non-empty cycle. Finally, $(A,\tau_f^{(n)})$ is separable iff (A,f) is countably generated.*

Now we describe unary algebras of a certain special form. The following definition is a modification of definition 2.4. in [8], where there are considered algebras with $A_f^{\infty_1} \neq \phi$.

2.2. Definition. Let (A,f) be a connected unary algebra with the property $R(A,f) = 1$, let (B,g) be a connected unary algebra such that $A \cap B = \phi$. Let $c \in B_g^0$. Then $(A,f) \oplus_c (B,g)$ denotes the unary algebra (C,h) defined as follows: $C = B \cup (A - A_f^{\infty_2})$ and for each $x \in C$ it is

$$h(x) = \begin{cases} f(x) & \text{for } x \in A - (A_f^{\infty_2} \cup f^{-1}(A_f^{\infty_2})) \\ c & \text{for } x \in f^{-1}(A_f^{\infty_2}) - A_f^{\infty_2} \\ g(x) & \text{for } x \in B \quad . \end{cases}$$

2.3. Definition. A connected unary algebra (A,f) is said to be reduced if it has one of the following forms:

(i) $R(A,f) = 1$ and $f^2 = f$.

(ii) Either $A = A_f^{\infty_1}$ or $A = A_f^{\infty_1} \cup A_f^0$, where $(A_f^{\infty_1}, \leqslant_f)$ is a chain of the type $\omega_0^* \oplus \omega_0$ and $A_f^0 \neq \phi$.

(iii) $(A,f) = (A_1,f_1) \oplus_c (A_2,f_2)$, where f_1 is a constant mapping and $(A_2, \leqslant_{f_2})$ is a chain of the type ω_0 with the first element c.

Remark. It is easy to see that $(A,\leqslant_f)$ is a chain of the type $\omega_0^* \oplus \omega_0$ if and only if f is a connected permutation of a countable set. Such an algebra (A,f) is also called a two-way infinite chain (see [16]). The algebra (A,f), where $(A,\leqslant_f)$ is a chain of the type ω_0 with the first element c is called according to [16] a one-way infinite chain with the generator c.

3. Analyzing various types of connected unary algebras we obtain the following results.

3.1. Proposition. *Let* (A,f) *be a connected unary algebra having the rank* $R(A,f) \geqslant 3$. *Then* $C(f) \neq S(A,\tau)$ *for every quasi-discrete topology* τ *on the set* A .

3.2. Theorem. *Let* (A,f) *be a connected unary algebra. The following conditions are equivalent:*

1) (A,f) *is either reduced or* A *is a cycle of the rank* 2.
2) $C(f) = S(A,\tau_f)$.
3) *There exists a quasi-discrete topology* τ *on* A *such that* $C(f) = S(A,\tau)$.

If we demand the quasi-discrete space (A,τ) to be also a T_0-space (i.e. a discrete space of Alexandroff), then we get the following theorem similar to 3.2.

3.3. Theorem. *Let* (A,f) *be a connected unary algebra. The following conditions are equivalent.*

1) (A,f) *is reduced.*
2) $C(f) = S(A,\tau_f)$.
3) *There exists a discrete topology of Alexandroff* τ *on the set* A *such that* $C(f) = S(A,\tau)$.

The answer to the question about unicity follows from the following proposition, but there seems to be open the problem of the description of all quasi-discrete topologies τ on A such that $S(A,\tau) = C(f)$ holds for a suitable f.

3.4. Proposition. *Let* (A,f) *be a connected reduced unary algebra. Then* $C(f) = S(A,\tau_f^{(n)})$ *for all positive integers.*

4. Now consider set transformations, disconnected in general, centralizers of which are formed by closed deformations of quasi-discrete topological spaces. Using Theorems 3.2 and 3.3 we get the following results. For the sake of brevity we write $A_\iota^{\infty_1}$, $A_\iota^{\infty_2}$, A_ι^0 instead of $A_{f_\iota}^{\infty_1}$, $A_{f_\iota}^{\infty_2}$, $A_{f_\iota}^0$ respectively.

4.1. Theorem. *Let* $(A,f) = \sum_{\iota \in J} (A_\iota, f_\iota)$ *be a unary algebra,* $J_0 = \{ \iota \in J : \operatorname{card} A_\iota > 1 \}$. *There exists a quasi-discrete topology* τ *on the set* A *with the property* $C(f) = S(A,\tau)$ *iff exactly one of the following conditions is satisfied:*

1) $\iota \in J$ *implies that either* A_ι *is a two-element cycle or* (A_ι, f_ι) *is idempotent.*

2) $\iota \in J_0$ *implies* $A_\iota = A_\iota^{\infty_1}$.

3) $\iota \in J_0$ *implies that either* (A_ι, f_ι) *is a two-way infinite chain or* $A_\iota = A_\iota^{\infty_1} \cup$ $\cup A_\iota^0$, *where* $(A_\iota^{\infty_1}, f_\iota)$ *is a two-way infinite chain and* $A_\iota^0 \neq \phi$.

4) $\iota \in J_0$ *implies that either* (A_ι, f_ι) *is a two-way infinite chain or* $(A_\iota, f_\iota) =$ $= (B_\iota, g_\iota) \oplus_c (C_\iota, h_\iota)$, *where* (B_ι, g_ι) *is an idempotent connected unary algebra with* $\operatorname{card} B \geqslant 2$ *and* (C_ι, h_ι) *is a one-way infinite chain with the generator* c .

Remark. If we require in theorem 4.1. the topology τ to be also a T_0-topology, we get a similar theorem with the difference only that two-element cycles are not admissible.

Theorem 4.1 implies

4.2. Proposition. *Let* $(A,f) = \sum_{\iota \in J} (A_\iota, f_\iota)$ *be a unary algebra. The following conditions are equivalent:*

1) *There exists a quasi-discrete topological space* (A,τ) *with the fixed point property for continuous closed mappings such that* $C(f) = S(A,\tau)$.

2) *There exists a connected compact discrete topology of Alexandroff* τ *on the set* A *with* $C(f) = S(A,\tau)$.

3) $\operatorname{card}[\{\iota \in J : f_\iota^2 = f_\iota, \operatorname{card} A_\iota \geqslant 2\} \cup \{\iota \in J : \operatorname{card} A_\iota = 1\}] = 1$. *Such a space* (A,τ) *is unique and the system of all non-void closed sets in that space is formed by the principal filter generated by the fixed point of the mapping* f .

4.3. Proposition. *Let* f *be a transformation of a set* A . *There exists a compact discrete topology of Alexandroff* τ *on the set* A *with* $C(f) = S(A,\tau)$ *iff the mapping* f *is idempotent with a finite set of fixed points. Such a space* (A,τ) *is unique and* $\tau X = X \cup f(X)$ *for each* $X \subseteq A$.

If we restrict our considerations to unary algebras with finite carrier sets, we get the corresponding results for autonomous automata. An autonomous automaton is a mapping f of a finite set $D(f)$ into itself (see [15]) . More detail about realizations of autonomous automata in this sense by closure operations can be found in [4]. An autonomous automaton f is called a permutation if f^k is the identity map, for a positive integer k ; the smallest positive k with this property is called the period of f, f is called a tree if f^k is a constant map for a non-negative integer k .

The smallest k with this property is called the height of f. Denoting by $E(f)$ the endomorphism monoid of the autonomous automaton f (using the terminology of paper [15]), we get from theorem 4.1 :

4.4. Proposition. *Let f be an autonomous automaton. There exists a topology τ on the set $D(f)$ such that $E(f) = S(D(f), \tau)$ iff the automaton f is the sum of trees of the height 1 and of permutations having periods at most 2.*

Proofs and other details will be published in [4], [5] and [6].

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