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TOPOLOGIES ON SYSTEMS OF SUBSETS

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1. Preliminaries.

Having a general topological space $(X,\mathcal{T})$ ($\mathcal{T}$ - the system of all open sets) $\mathrm{Exp}(X,\mathcal{T})$ denotes the system of all nonempty subsets of X, $2^{(X,\mathcal{T})}$ - the system of all nonempty closed subsets in $(X,\mathcal{T})$. For $\mathrm{Exp}(X,\mathcal{T})$ and $2^{(X,\mathcal{T})}$ many different topologies were defined in the past. Now, we shall be interested in the most classical ones. Let $K(X,\mathcal{T})$ denote one of the system $\mathrm{Exp}(X,\mathcal{T})$ or $2^{(X,\mathcal{T})}$. Put $\mathcal{S}_1(X,\mathcal{T}) = \{\{A : A \in K(X,\mathcal{T}), A \subset O_1 \cup \dots \cup O_n, A \cap O_i \neq \phi$ for $i \in \{1,\dots,n\}\}$: : n - positive integer, $O_1, \dots, O_n \in \mathcal{T}\}$.
$\mathcal{S}_2(X,\mathcal{T}) = \{\{A : A \in K(X,\mathcal{T}), A \cap O_i \neq \phi$ for $i = \{1, \dots, n\}\} : n$ - positive integer, $O_1, \dots, O_n \in \in \mathcal{T}\}$. $\mathcal{S}_3(X,\mathcal{T}) = \{\{A : A \in K(X,\mathcal{T}), A \subset O\} : O \in \mathcal{T}\}$. $\mathcal{S}_i(X,\mathcal{T})$ is a base for the topology $\mathcal{T}_i(X,\mathcal{T})$ on $K(X,\mathcal{T})$ (called for $i = 1$ finite or Vietoris topology, for $i = 2$ lower semifinite topology, for $i = 3$ upper semifinite topology).

$\mathbf{T}_i(X,\mathcal{T})$ denotes the corresponding topological space.

We shall discuss certain possibilities of extension of the "object" function given for fixed i and fixed $K(X,\mathcal{T})$ by $(X,\mathcal{T}) \to \mathbf{T}_i(X,\mathcal{T})$ to a functor.

If $f : (X,\mathcal{T}) \to (Y,\mathcal{T}')$ is continuous, one defines $f^* : K(X,\mathcal{T}) \to K(Y,\mathcal{T}')$ by $f^*(A) = \{f(a) : a \in A\}$ for $K(X,\mathcal{T}) = \mathrm{Exp}(X,\mathcal{T})$ and $f^*(A) = \overline{\{f(a) : a \in A\}}$ for $K(X,\mathcal{T}) = 2^{(X,\mathcal{T})}$ ($\overline{}$ means closure in $\mathcal{T}'$). Put $\mathbf{T}_i(f) = f^*$. Denote by T the category of all Bourbaki topological spaces, by T_0 the category of all T_0-spaces, by T_1 – T_1-spaces (allways with continuous mappings as morphisms).

2. The case $\mathrm{Exp}(X,\mathcal{T})$.

In this section $\mathbf{T}_1(X,\mathcal{T})$, $\mathbf{T}_2(X,\mathcal{T})$, $\mathbf{T}_3(X,\mathcal{T})$, f^* are defined with respect to $\mathrm{Exp}(X,\mathcal{T})$.

Proposition 1. a) *For* $i = 1,2,3$ $\mathbf{T}_i(X,\mathcal{T})$, $\mathbf{T}_i(f)$ *yield a faithful functor from* T *to* T.
b) *For* $i = 1,2,$ $\mathbf{T}_i(X,\mathcal{T})$, $\mathbf{T}_i(f)$ *yield a faithful functor from* T_0 *to* T_0.
c) $\mathbf{T}_3(X,\mathcal{T})$, $\mathbf{T}_3(f)$ *yield a faithful functor from* T_1 *to* T_0.

P r o o f. Ad a) The only not immediately clear fact to be proved is the continuity of $\mathbf{T}_i(f)$. For $i = 1$ the proof goes along the lines of the proof of 5.10.1 in [5], for $i = 2,3$ the proofs are quite analogous.

Ad b) For $i = 1$ se [5], for $i = 2$ see 2.2 in [1].

Ad c) See [4].

Now, let us check the continuity of the following mappings, which will be of importance in section

4 . Take $(X,\mathcal{T}) \in T$ and put $\eta : X \to \mathrm{Exp}(X,\mathcal{T})$ with $\eta(x) = \{x\}$ for $x \in X$.
$\mu : \mathrm{Exp}(\mathbf{T}_i(X,\mathcal{T})) \to \mathrm{Exp}(X,\mathcal{T})$ with $\mu(A) = \bigcup_{A \in A} A$ for $A \in \mathrm{Exp}(\mathbf{T}_i(X,\mathcal{T}))$ (so μ is so called union mapping denoted in [5] as σ) .

Proposition 2. η and μ *are continuous.*

P r o o f. For η the proof is straightforward (see [5] p.153 for $i = 1$) , for μ and $i = 1$ see [5] 5.7.2 , for $i = 2,3$ the proofs are analogous. □

3. The case $2^{(X,\mathcal{T})}$

In this section $\mathbf{T}_1(X,\mathcal{T})$, $\mathbf{T}_2(X,\mathcal{T})$, $\mathbf{T}_3(X,\mathcal{T})$, f^* are defined with respect to $2^{(X,\mathcal{T})}$.

Proposition 3. *Let* $(X,\mathcal{T})$ *be a topological space. The following assertions are equivalent:*

A) $(X,\mathcal{T})$ *is normal (i.e. two disjoint closed subsets in* $(X,\mathcal{T})$ *are always separated).*

B) *For every topological space* $(Y,\mathcal{T}')$ *and every continuous mapping* $f : (Y,\mathcal{T}') \to (X,\mathcal{T})$ *the mapping* $f^* : \mathbf{T}_1(Y,\mathcal{T}') \to \mathbf{T}_1(X,\mathcal{T})$ *is continuous.*

P r o o f. Suppose $(X,\mathcal{T})$ is normal. Put $\langle O_1, \dots, O_n \rangle = \{A : A \in 2^{(X,\mathcal{T})}, A \subset O_1 \cup \dots \cup O_n , A \cap O_i \neq \phi$ for $i = 1, \dots, n\}$ for $O_1, \dots, O_n \in \mathcal{T}$ (similar notation will be used for all spaces) . Let $A \in 2^{(Y,\mathcal{T}')}$, $\langle O_1, \dots, O_n \rangle$ be a neighborhood of $\overline{f(A)}$ in $\mathbf{T}_1(X,\mathcal{T})$. Take such open O in $\mathcal{T}$ for which $\overline{f(A)} \subset O \subset \overline{O} \subset O_1 \cup \dots \cup O_n$. Put $O'_i = O \cap O_i$. $\langle f^{-1}(O'_1), \dots, f^{-1}(O'_n) \rangle$ is a neighborhood of A in $\mathbf{T}_1(Y,\mathcal{T}')$ $(A \cap f^{-1}(O'_i) = \phi \Rightarrow f(A) \cap O'_i = \phi \Rightarrow \overline{f(A)} \cap O'_i = \phi$, a contradiction) . Let $B \in \langle f^{-1}(O'_1), \dots, f^{-1}(O'_n) \rangle$, i.e. $B \subset \bigcup_{i=1}^{n} f^{-1}(O'_i)$. Then $f(B) \subset \bigcup_{i=1}^{n} O'_i \subset O$ and $\overline{f(B)} \subset \overline{O} \subset O_1 \cup \dots \cup O_n$. It is $\overline{f(B)} \cap O_i \neq \phi$ as $B \cap f^{-1}(O'_i) \neq \phi$ implies $f(B) \cap O_i \neq \phi$. So $f^*(\langle f^{-1}(O'_1), \dots, f^{-1}(O'_n) \rangle) \subset \langle O_1, \dots, O_n \rangle$.

Suppose $(X,\mathcal{T})$ is not normal. Let O be a neighborhood of a closed set M in $(X,\mathcal{T})$ such that $\overline{U}$ non $\subset O$ for any neighborhood U of M . Let $(Y,\mathcal{T}')$ be the topological space defined as follows.

1. $Y = O$. 2. $x \in O - M$ is isolated. 3. $V \subset O$, $V \cap M \neq \phi$ is open in $(Y,\mathcal{T}')$ iff there is V' open in $(X,\mathcal{T})$ such that $V' \subset V$ and $V' \cap M = V \cap M$. The inclusion map i from Y to X is clearly continuous. Now, $\langle O \rangle$ is a neighborhood of M in $\mathbf{T}_1(X,\mathcal{T})$. Suppose there is some open set $\langle O_1, \dots, O_n \rangle$ in $\mathbf{T}_1(Y,\mathcal{T}')$ containing M , for which $i^*(\langle O_1, \dots, O_n \rangle) \subset \langle O \rangle$. We have $M \subset O_1 \cup \dots \cup O_n$, and there is such $O' \supset M$, which is open in $(X,\mathcal{T})$ and $O' \subset O_1 \cup \dots \cup O_n$ The set O' is closed in $(Y,\mathcal{T}')$, therefore $O' \in \langle O_1, \dots, O_n \rangle$ as clearly $O' \cap O_i \neq \phi$. We should have $i^*(O') \in \langle O \rangle$. But $i^*(O') = \overline{O'}$ (closure in $(X,\mathcal{T})$) and so we would have $O' \subset O$, which is impossible. □

Remark. Proposition 3 suggests that it is reasonable to restrict oneself at least to normal spaces if one wants to get a category where $\mathbf{T}_1(X,\mathcal{T})$, $\mathbf{T}_1(f)$ yield an endofunctor. As in [2] Keesling under CH and Veličko in [8] without CH proved that normality of $\mathbf{T}_1(X,\mathcal{T})$ implies compactness of $(X,\mathcal{T})$, the restriction goes to compact spaces, where really such an endofunctor exists.

Proposition 4. $\mathbf{T}_2(X,\mathcal{T})$, $\mathbf{T}_2(f)$ *yield an endofunctor in* $\boldsymbol{T}$.

P r o o f. As $\overline{f(\overline{g(A)})} = \overline{fg(A)}$ for continuous mappings f and g, it is sufficient to prove that for continuous f, $f: (X,\mathcal{T}') \to (Y,\mathcal{T}'')$ the map f^* is continuous. Let A be closed in $(X,\mathcal{T}')$ and $f^*(A) = \overline{f(A)}$. Let $O_1, \ldots, O_n$ be open sets in $(Y,\mathcal{T}'')$ defining the open set $\boldsymbol{O}$ in $\mathbf{T}_2(Y,\mathcal{T}'')$. Suppose $f^*(A) \in \boldsymbol{O}$. The set $f^{-1}(O_i)$ is open in $(X,\mathcal{T}')$ and all these open sets define the open set $\boldsymbol{O}'$ in $\mathbf{T}_2(X,\mathcal{T}')$. Let $B \in \boldsymbol{O}$. Clearly $\overline{f(B)} \cap O_i \neq \phi$ for all i. Therefore $f^*(\boldsymbol{O}') \subset \boldsymbol{O}$. In the same time, $\boldsymbol{O}'$ is a neighborhood of A as $A \cap f^{-1}(O_i) = \phi$ implies $\overline{f(A)} \cap O_i = \phi$. □

Similarly to the demonstration of Proposition 3 one proves

Proposition 5. *The assertion of Proposition* 3 *is valid after replacement* $\mathbf{T}_1$ *for* $\mathbf{T}_3$.

Lemma. *Every nonempty closed set in* $\mathbf{T}_3(X,\mathcal{T})$ *contains* X.

P r o o f follows from the fact that open sets in $\mathbf{T}_3(X,\mathcal{T})$ are hereditary with respect to closed subsets (in $(X,\mathcal{T})$). □

Corollary 1. $\mathbf{T}_3(X,\mathcal{T})$ is always a normal space.

Corollary 2. Let T_4 be the category of all normal spaces. $\mathbf{T}_3(X,\mathcal{T})$, $\mathbf{T}_3(f)$ yield an endofunctor in T_4.

4. Algebras for $\mathbf{T}_1$ in the case $\mathrm{Exp}(X,\mathcal{T})$.

Results of the section 2 imply that $\mathbf{T}_i$ $(i = 1,2,3)$ defined with respect to $\mathrm{Exp}(X,\mathcal{T})$ with η,μ is a nomad in T ([3]). We shall make some remarks on algebras for $\mathbf{T}_1$. By result of Manes proved in his Thesis for the category of sets these algebras are complete upper semilattices $(X,\sup)$ together with certain topology $\mathcal{T}$ on X. From continuity of multiplication in algebra for $\mathbf{T}_1$ one gets the following necessary and sufficient condition on sup and $\mathcal{T}$ to have such an algebra:

If A is a nonempty subset of X, $a = \sup A$ and O a neighborhood of a, then there exist open sets $O_1, \ldots, O_n$ in $\mathcal{T}$ such that

1. $A \subset O_1 \cup \ldots \cup O_n$.
2. $A \cap O_i \neq \phi$ for all i.
3. When $B \subset X$, $B \subset O_1 \cup \ldots \cup O_n$, $B \cap O_i \neq \phi$ for all i then $\sup B \in O$.

A complete upper semilattice with such a topology will be called a $\mathbf{T}_1$-semilattice and denoted $(X,\sup,\mathcal{T})$.

Proposition 6. *If* $(X,\sup,\mathcal{T})$ *is a* $\mathbf{T}_1$*-semilattice and* $\mathcal{T}$ *is Hausdorff then* $\mathcal{T}$ *is compatible with the ordering* $\leqslant$ *of* $(X,\sup)$ *in the sense of* [6] *(see* [7], *too)*.

P r o o f . Let $a,b \in X$, $a < b$. Take disjoint open sets $O',O'', a \in O'$, $b \in O''$. It is easy to see, that there exist disjoint open sets O_1,O_2 with $a \in O_1$, $b \in O_2$, $O_1 \cap O'' = \phi$ and such that for $x \in$ $\in O_1$, $y \in O_2$ we have $\sup\{x,y\} \in O''$. For $x \in O_1$, $x > b$ implies $x \in O''$; $a > y$ for some $y \in O_2$ gives $a \in O''$. Both conclusions are false. □

Example. Take a chain of the type $(\omega + 2)^*$, say $M = \{a_0,a_1, \dots , b_n, \dots , b_0\}$, $a_0 < a_1 < \dots$ $\dots < b_n \dots < b_0$. Define the topology $\mathcal{T}$ on M in the following way: all b_i are isolated, a_0,a_1 are contained only in open sets with the finite complement. $\mathcal{T}$ is T_1-topology, $(M,\sup,\mathcal{T})$ is $\mathbf{T}_1$-semilattice and every neighborhood of a_0 containts points greater then a_1. So in this case $\mathcal{T}$ is not compatible with the ordering.

The coarsest Kuratowski topology $\mathcal{T}$ on X has only ϕ,X and finite sets for open sets.

Proposition 7. *Let* $(X,\sup,\mathcal{T})$ *be a* $\mathbf{T}_1$*-semilattice with infinite* X. *Then* $\mathcal{T}$ *is not the coarsest Kuratowski topology.*

P r o o f . The main tool si the following simple lemma. □

Lemma. *Let* $(X,\leqslant)$ *be an infinite complete upper semilattice. Then at least one of the following assertions is true:*

1. *There exists* $a \in X$ *having infinitely many neighbors under itself.*
2. *There exists a chain of the type* ω *in* $(X,\leqslant)$.
3. *There exists a chain of the type* ω^* *in* $(X,\leqslant)$.

So let $(X,\sup,\mathcal{T})$ be an infinite $\mathbf{T}_1$-semilattice. Suppose $\mathcal{T}$ is the coarsest Kuratowski topology. Let 1. be valid from lemma. Then the element a should be contained in every open set as every open set contains a',a'' such that $a = \sup(a',a'')$. This is a contradiction.

If $a_0 < a_1 < a_2 < \dots$ is a chain in $(X,\sup)$, then every open set in $\mathcal{T}$ contains almost all elements a_i and also $a = \sup\{a_0,a_1, \dots\}$. If $a_0 > a_1 > a_2 > \dots$ is a chain in $(X,\sup)$, take $O \in \mathcal{T}$, $a_0 \in O$, a_1 non $\in O$ Now $(X,\sup,\mathcal{T})$ is a $\mathbf{T}_1$-semilattice. Let $O_1, \dots , O_n \in \mathcal{T}$, $\{a_0,a_1,a_2, \dots\} \subset$ $\subset O_1 \cup \dots \cup O_n$ such that $A \subset \{a_0,a_1,a_2, \dots\}$, $A \cap O_i \neq \phi$ implies $\sup A \in O$. We can put $A =$ $= \{a_1,a_2,a_3, \dots\}$. Then $\sup A = a_1 \in O$, a contradiction.

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