Karel Wichterle

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JOINT-CONVERGENCE IN FUNCTION SPACES. ORDER OF $\mathcal{W}$-CLOSURES.

K.WICHTERLE

PRAHA

It is well-known (cf. [4]) that if $\langle X,u\rangle$ is a (sequential) convergence space, then u^{ω_1} is a topology for X and there are examples such that u^{ξ} fails to be a topology for each ordinal $\xi < \omega_1$. In the first part of the present paper we generalize these results to $\mathcal{W}$-convergence spaces (cf. [5]). In the second part we introduce joint-convergences $\mathcal{C}^j$ and $\mathcal{C}^k$ on the set of all mappings on a set into a closure space, study their properties and their relations to pointwise, continuous and uniform convergences. Further, a characterization of sequentially compact uniformizable spaces is given.

I

First we recall some definitions. Let $\mathcal{W}$ be a class of directed sets and $\langle X,u\rangle$ a closure space. A $\mathcal{W}$-net is a net with domain in $\mathcal{W}$ and u is a $\mathcal{W}$-closure if it is determined by the convergence of $\mathcal{W}$-nets ranging in X (cf. [5]). For each ordinal ξ we define a closure u^{ξ} as follows: $u^0 A = A$, $u^{\xi} A = u(u^{\xi-1}A)$ if ξ is isolated and $u^{\xi}A = \bigcup\{u^{\eta}A;\ \eta<\xi\}$ if ξ is a limit ordinal. The order of u is the least ordinal ξ such that for each $A\subset X$ we have $u^{\xi+1}A = u^{\xi}A$.

Let D and E be directed sets. We say (cf. [3]) that D is a quotient of E, in symbols $D\prec E$, if there is a convergent mapping on E into D. If $D\prec E$ and $E\prec D$, then we say that D and E have the same cofinal type $\mathrm{cf}D = \mathrm{cf}E$. If $D\prec E$, then we write $\mathrm{cf}D \precsim \mathrm{cf}E$. Denote by $Q[\mathcal{W}]$ the class of all regular cardinals which are quotients of elements of $\mathcal{W}$.

Theorem 1. Let Careg be the class of all regular cardinals. Then:

(a) Let $Q[\mathcal{W}]$ = Careg. Then for every ordinal ξ there exists a $\mathcal{W}$-closure w with order ξ.

(b) Let $Q[\mathcal{W}] \neq$ Careg and $\beta = \mathrm{Min}(\mathrm{Careg} - Q[\mathcal{W}])$. Then u^{β} is the topological modification of u for every $\mathcal{W}$-closure u. Further, there exist a $\mathcal{W}$-closure v and a set Y such that $v^{\xi}Y \neq v^{\beta}Y$ for each $\xi < \beta$.

Proof. (a) Let $\alpha = \mathrm{Min}(Q[\mathcal{W}])$, $P = \alpha \cup \{\alpha\}$, and $X = \{x = \{x_{\eta} \mid \eta < \xi\} \in P^{\xi};\ \vartheta < \eta < \xi \Rightarrow (x_{\eta} = \alpha \Rightarrow x_{\vartheta} = \alpha)\}$. For each $x\in X$ define the base B_x of w-neighborhoods of x as follows:

Put $\gamma_x = \xi$ if $x = \{\alpha \mid \eta < \xi\}$ and $\gamma_x = \mathrm{Min}(\{\eta < \xi\,;\ x_\eta \neq \alpha\})$ otherwise. If $\gamma_x = \varkappa + 1$, then $B_x = \{\{y \in X;\ y_\varkappa > \vartheta \text{ and } (\eta \neq \varkappa \Rightarrow y_\eta = x_\eta)\}\,;\ \vartheta < \alpha\}$. If γ_x is not isolated, then $B_x = \{\{y \in X;\ (\eta \geq \gamma_x \text{ or } \eta < \vartheta) \Rightarrow$ $\Rightarrow y_\eta = x_\eta\}\,;\ \vartheta < \gamma_x\}$.

For each $x \in X$, the space $\mathfrak{X}_\xi = \langle X, w\rangle$ has a monotone local base at x with the cofinal type α or cf γ_x . Thus w is a $\mathcal{W}$-closure, because α and all cf γ_x are elements of $Q[\mathcal{W}]$ = Careg . For the set $Y =$ $= \{y \in X;\ \eta < \xi \Rightarrow y_\eta < \alpha\}$ we have $u^{\vartheta} Y = \{y \in X;\ \eta \geq \vartheta \Rightarrow y_\eta < \alpha\}$.

(b) Let be $x \in u(u^{\beta} Z)$. Then some $\mathcal{W}$-net $\{x_a \mid a \in D\}$ ranging in $u^{\beta} Z$ converges to x . For each $a \in D$ denote $\varphi\, a = \mathrm{Min}(\{\eta < \beta;\ x_a \in u^{\eta} Z\})$. Because $D \in \mathcal{W}$ and $\beta \notin Q[\mathcal{W}]$, the mapping φ from D into β is not convergent. Hence for some ordinal $\xi < \beta$ the set $E = \{a \in D;\ \varphi a < \xi\}$ is cofinal in D. Then $x_a \in u^{\xi} Z$ for each $a \in E$ and $x \in u(u^{\xi} Z) \subset u^{\beta} Z$.

Further, we define $\langle P, v\rangle$ as the sum of closure spaces $\{\mathfrak{X}_\xi \mid \xi < \beta\}$ from (a). Then v is a $\mathcal{W}$-closure, because for each $x \in P$ we have cf $\gamma_x \leq \gamma_x \leq \xi < \beta$, and hence cf $\gamma_x \in Q[\mathcal{W}]$.

<u>Remark</u>. The closures v and u are functionally separated, chain-net-closures, and for their cardinalities and local characters χ^L (cf. [1] p.260) we have: $\chi^L(w) = \chi^L(\mathfrak{X}_\xi) = \omega_0 . \mathrm{card}\, \xi$, card $X_\xi = \omega_0 . 2^{\xi}$, $\chi^L(v) < \alpha^+ . \beta$, card $P = \alpha . 2^{\beta}$ (card $P = \alpha . \beta$, if GCH holds) .

<u>Remark</u>. Let $\mathcal{W} = \{\omega_0\}$. Then $\mathcal{W}$-nets are sequences, $\mathcal{W}$-closures are (sequential) convergence closures (cf. [4]) , $Q[\mathcal{W}] = \{\omega_0\}$, Min (Careg - $Q[\mathcal{W}]$) = ω_1 ; therefore the order of every $\mathcal{W}$-closure is at most ω_1 . Consequently, Theorem 1 generalizes the results mentioned in the introduction.

II

In this section we shall define and study joint-convergences $\mathcal{C}^j(\mathfrak{X}^T)$ and $\mathcal{C}^k(\mathfrak{X}^T)$ on the set X^T of all mappings on a set T into a closure space $\mathfrak{X} = \langle X, u\rangle$. First we introduce two auxiliary convergences.

<u>Definition 1</u>. Let $f \in X^T$ and let $N = \{f_a \mid a \in D\}$ be a net ranging in X^T, We say that <u>N 1-converges to f</u> if the following implication holds true:

If $t \in T$ and $\{t_b \mid b \in B\}$ is a net ranging in T such that for each $a \in D$

the net $\{f_a t_b \mid b \in B\}$ converges to $f_a t$, then the double net $\{f_a t_b \mid \langle a,b\rangle \in D \times B\}$ converges to ft (in $\mathfrak{X}$).

We say that <u>N 2-converges to f</u> if the following implication holds true: If $\{z_a \mid a \in D\}$ is a net ranging in X which converges to a point z in $\mathfrak{X}$ and $\{t_b \mid b \in B\}$ is a net ranging in T such that for each $a \in D$ the net $\{f_a t_b \mid b \in B\}$ converges to z_a, then the double net $\{f_a t_b \mid \langle a,b\rangle \in D \times B\}$ converges to z.

<u>Remark</u>. The definition of a 1-convergence coincides with the definition of a continuous convergence given by Z.Frolík in [2]. Notice that 1-convergence and 2-convergence need not determine a closure for X^T (see Example 1). To avoid this, we shall introduce joint-convergences $\mathcal{C}^j$ and $\mathcal{C}^k$. However, if we restrict ourselves to decreasing sequences converging to constant mappings (cf. [2]), then 1-convergence and $\mathcal{C}^j$ coincide and determine a closure.

<u>Definition 2</u>. Let $f \in X^T$ and let N be a net ranging in X^T. We say that <u>N $\mathcal{C}^j$-converges to f</u> if every subnet of N 1-converges to f. We say that <u>N $\mathcal{C}^k$-converges to f</u> if every subnet of N 2-converges to f or N is eventually equal to f.

<u>Notation</u>. Denote by $\mathcal{C}^j = \mathcal{C}^j(\mathfrak{X}^T)$, resp. $\mathcal{C}^k = \mathcal{C}^k(\mathfrak{X}^T)$, the class of all pairs $\langle N,f\rangle$ such that N $\mathcal{C}^j$-converges, resp. $\mathcal{C}^k$-converges, to f. Denote by $\mathcal{C}_p$ the pointwise convergence on X^T.

<u>Remark</u>. If $\mathfrak{X}$ is topological or separated, then the condition "N is eventually equal to f" implies that all subnets of N 2-converge and therefore can be omitted.

If card $X = 1$, then $\mathcal{C}^j$ and $\mathcal{C}^k$ are trivial. If card $X > 1$, then there are examples such that $\mathcal{C}^j$ and $\mathcal{C}^k$ are non-trivial.

<u>Proposition</u>. We have $\mathcal{C}^k \subset \mathcal{C}^j \subset \mathcal{C}_p$. If T is finite, then $\mathcal{C}^k = \mathcal{C}^j = \mathcal{C}_p$.

<u>Proposition</u>. $\langle N,f\rangle \in \mathcal{C}^j$ iff every generalized subnet of N 1-converges to f.

<u>Remark</u>. There exist a space $\mathfrak{X}$ and $\langle N,f\rangle \in \mathcal{C}^k(\mathfrak{X}^T)$ such that not every generalized subnet of N 2-converges.

<u>Proposition</u>. $\mathcal{C}^j$ and $\mathcal{C}^k$ are convergence structures and fulfil the Urysohn's axiom.

<u>Remark</u>. It is an open problem, whether or not $\mathcal{C}^j$ and $\mathcal{C}^k$ are convergence classes (cf. [1]).

<u>Notation</u>. From the above Proposition it follows (cf. [1]) that $\mathcal{C}^j$ and $\mathcal{C}^k$ determine closures. Denote them u_j, resp. u_k.

<u>Proposition</u>. Let $i \in \{1,2\}$. Then the following are equivalent:

(a) $\mathcal{X}$ is a T_i-space.

(b) $\langle X^T, u_j \rangle$ is a T_i-space.

(c) $\langle X^T, u_k \rangle$ is a T_i-space.

<u>Proposition</u>. Let $N = \{f_a \mid a \in A\}$ be a net ranging in X^T and let $\langle N, f \rangle \in \mathcal{C}^j(\mathcal{X}^T)$. Let v be a closure for T such that $\{a; f_a$ is continuous$\}$ is a residual subset of A. Then N converges continuously to f.

The following example shows that the converse implication is false.

<u>Example 1</u>. Let $T = \omega_o \cup \{r,s\}$, let a closure space $\mathcal{X}$ contains at least three closed points x, y, and z. Define a net $\{f_n \mid n \in \omega_o\}$ (ranging in X^T) and $f \in X^T$ as follows: $fk = f_n k = x$ for $n \geq k$; $f_n k = y$ for $n < k$, n is odd; $f_n k = z$ for $n < k$, n is even; $f_n r = fr = y$; $f_n s = fs = z$. Let $G = \{f_n \mid n$ is odd$\}$ and $H = \{f_n \mid n$ is even$\}$. Let u_i ($i \in \{1,2\}$) be a mapping on the power set of X^T defined by $u_i A = \{g \in X^T$; there is a net ranging in A which i-converges to $g\}$. Then N i-converges to f, $f \in u_i(G \cup H) - (u_i G \cup u_i H)$, $\langle N, f \rangle \notin \mathcal{C}^j$, and u_i is not a closure for X^T. If v is a closure for T such that the set $\{n; f_n$ is continuous$\}$ is residual in ω_o, then r and s are isolated in $\langle T, v \rangle$ and N converges continuously to f.

<u>Proposition</u>. Let $\mathcal{X}$ be a partially ordered sequentially compact topological T_2-space such that each point in $\mathcal{X}$ has a base of interval-like neighborhoods. Let N be a decreasing net which 2-converges to f. Then $\langle N, f \rangle \in \mathcal{C}^k(\mathcal{X}^T)$.

<u>Remark</u>. A counterexample shows that in the above Proposition the 2-convergence and $\mathcal{C}^k$ cannot be replaced by the 1-convergence and $\mathcal{C}^j$ even if $\mathcal{X}$ is a bounded interval.

Definition 3. Let $\mathcal{W}$ be a class of directed sets. We define classes $\mathcal{C}^j_{\mathcal{W}} = \mathcal{C}^j_{\mathcal{W}}(\mathfrak{X}^T)$, resp. $\mathcal{C}^k_{\mathcal{W}} = \mathcal{C}^k_{\mathcal{W}}(\mathfrak{X}^T)$, in the some way as classes $\mathcal{C}^j$, resp. $\mathcal{C}^k$, provided that in Definition 1 we assume that the nets $\{t_b \mid b \in B\}$ are $\mathcal{W}$-nets. Further, for $\mathcal{W} = \{\omega_0\}$ we put $\mathcal{C}^j_{\{\omega_0\}} = \mathcal{C}^s$, $\mathcal{C}^k_{\{\omega_0\}} = \mathcal{C}^\sigma$.

Proposition. Let $\mathcal{W}$ and $\mathcal{W}'$ be classes of directed sets, $\mathcal{C}_p$ the pointwise convergence on X^T, and $i \in \{j,k\}$. Then:

(a) $\mathcal{W}' \subset \mathcal{W}$ implies $\mathcal{C}^i \subset \mathcal{C}^i_{\mathcal{W}} \subset \mathcal{C}^i_{\mathcal{W}'} \subset \mathcal{C}_p$.

(b) $\mathcal{C}^i_{\mathcal{W}} \neq \mathcal{C}_p$ if and only if $\operatorname{Min}(Q[\mathcal{W}]) \leq \operatorname{card} T$.

Notation. Denote by $\mathfrak{M}_\alpha$ the class of all directed sets $\langle E, \prec \rangle$ with card $E \leq \alpha$.

Proposition. Let $\mathcal{W}$ be a class of directed sets, $i \in \{j,k\}$ and card $T = \alpha$. Then the conditions (a), (b), and (c) below are equivalent and for all spaces $\mathfrak{X}$ (c) implies (d):

(a) There exists an $\mathfrak{M}_\alpha$-space which is not a $\mathcal{W}$-space.

(b) There exists a normal $\mathfrak{M}_\alpha$-space which is not a $\mathcal{W}$-space.

(c) The class $\operatorname{cf}[\mathcal{W}]$ is not cofinal in $\langle \operatorname{cf}[\mathcal{W} \cup \mathfrak{M}_\alpha], \prec\!\!\cdot \rangle$.

(d) $\mathcal{C}^i_{\mathcal{W}}(\mathfrak{X}^T) \neq \mathcal{C}^i(\mathfrak{X}^T)$.

The following example shows that the four conditions in the above Proposition are not equivalent.

Example 2. Let p be an ultrafilter on ω_0, $T = \omega_0 \cup \{s\}$, let x and y be two closed points in $\mathfrak{X}$ and $i \in \{j,k\}$. Define a net N $N = \{f_a \mid a \in p\}$ ranging in $\{x,y\}^T$ as follows: p is directed by the inverse inclusion $\supset$, $f_a n = y$ iff $n \in a$,($f_a n = x$ if $n \notin a$), and $f_a s = fs = y$. Then $\langle N,f \rangle \in \mathcal{C}^i_{\mathfrak{M}_{\omega_0}} - \mathcal{C}^i_{\{p\}} \subset \mathcal{C}^i_{\mathfrak{M}_{\omega_0}} - \mathcal{C}^i$. (For the proof of $\langle N,f \rangle \notin \mathcal{C}^i_{\{p\}}$ choose $t_b \in b$ for each $b \in p$.)

Theorem 2. Let be a first-countable topological space, let $N = \{f_d \mid d \in A\}$ be a net containing a subsequence, and let $\omega_0 \in Q[\mathcal{W}]$. Then the following are equivalent:

(a) $\langle N,f \rangle \in \mathcal{C}^j(\mathfrak{X}^T)$.

(b) $\langle N,f \rangle \in \mathcal{C}^s(\mathfrak{X}^T)$.

(c) $\langle N,f \rangle \in \mathcal{C}_p$ and the condition (b)holds in the following modified form: in Definition 2 subnets are replaced by subsequences and in the Definition 1 ($B=\omega_0$) the double sequence is replaced by the diagonal sequence.

(d) $\langle N,f\rangle \in \mathcal{C}^j_{\mathcal{V}}(\mathfrak{X}^T)$.

Remark. The analogous Theorem is true for $\mathcal{C}^k$ and $\mathcal{C}^\sigma$. If $\mathfrak{X}$ is discrete, then (c)can be simplyfied.

Now we shall consider the relations between $\mathcal{C}^k$ and the uniform convergence $\mathcal{C}_{\mathcal{U}}$ on the function space X^T.

Theorem 3. Let $\mathcal{U}$ be a uniformity inducing $\mathfrak{X}$. If a net N ranging in X^T converges $\mathcal{U}$-uniformly to $f\in X^T$, then $\langle N,f\rangle \in \mathcal{C}^k(\mathfrak{X}^T)$.

Proof. Let $\{f_a \mid a\in D\}$ be a subnet of N, $\{t_b \mid b\in B\}$ a net ranging in T, and $\{z_a \mid a\in A\}$ a net converging in $\mathfrak{X}$ to z such that the net $\{f_a t_b \mid b\in B\}$ converges to z_a for each $a\in D$. Let W be a neighborhood of z . Choose $U\in\mathcal{U}$ and $V\in\mathcal{U}$ such that $U[z]\subset W$ and $V\circ V\circ V\circ V\subset U$. Choose $d\in D$ such that $\langle z_d,z\rangle\in V$ and $\langle f_a t,ft\rangle\in V$ for each $t\in T$ and $a\succ d$. Choose $c\in B$ such that $\langle f_d t_b,z_d\rangle\in V$ for each $b\succ c$. If $a\succ d$ and $b\succ c$, then $\langle f_a t_b,ft_b\rangle\in V$ and $\langle ft_b,f_d t_b\rangle\in V$, and hence $\langle f_a t_b,z\rangle\in V\circ V\circ V\circ V\subset U$ and $f_a t_b\in W$.

Theorem 4. Let $\mathfrak{X}$ be a sequentially compact topological space, $\mathcal{U}$ a continuous uniformity for $\mathfrak{X}$. Let N be a net ranging in X^T which contains a subsequence. If $\langle N,f\rangle\in\mathcal{C}^\sigma(\mathfrak{X}^T)$, then N $\mathcal{U}$-uniformly converges to f .

Proof. Let $\langle N,f\rangle\in\mathcal{C}^\sigma-\mathcal{C}_{\mathcal{U}}$. Then we can find $U\in\mathcal{U}$, $V\in\mathcal{U}$ with $V\circ V\subset U$, a sequence $\{t_i \mid i\in\omega_0\}$ ranging in T, and a subsequence $\{g_i \mid i\in\omega_0\}$ of N such that (for each i,j satisfying $j<i<\omega_0$) $\langle g_i t_i,ft_i\rangle\notin U$ and $\langle g_i t_j,ft_j\rangle\in V$. Put $D_0=\omega_0$ and choose (inductively) $i_k\in D_k$ and $D_{k+1}\subset D_k$ such that sequences $\{g_{i_k} t_j \mid j\in D_{k+1}\}$ converge in $\mathfrak{X}$. Denote their limits by z_k , choose a convergent subsequence $\{z_k \mid k\in E\}$ and an open neighborhood G of its limit z . Then $\{g_{i_k} t_{i_j} \mid j\in\omega_0\}$ converges to z_k for each $k\in E$. Because $\langle N,f\rangle\in\mathcal{C}^\sigma$, the double net $\{g_{i_k} t_{i_j} \mid \langle k,j\rangle\in E\times\omega_0\}$ converges to z ; thus for large $k\in\omega_0$ $\langle g_{i_k} t_{i_k},g_{i_{k+1}} t_{i_k}\rangle\in G\times G\subset V$ and $\langle g_{i_k} t_{i_k},ft_{i_k}\rangle\in V\circ V-U$.

Theorem 5. Let $\mathcal{U}$ be a fine uniformity for $\mathfrak{X}$. If $\mathfrak{X}$ is not sequentially compact, then there exists a sequence N and a mapping f such that $\langle N,f\rangle \in \mathcal{C}^k(\mathfrak{X}^T)$ and N does not converge $\mathcal{U}$-uniformly to f.

Proof. We can find a sequence $\{y_n | n \in \omega_c\}$ without accumulation points and $U \in \mathcal{U}$ such that $n \neq m \Rightarrow \langle y_n, y_m\rangle \notin U$. We choose a bijective sequence $\{s_n | n \in \omega_c\}$ onto $S \subset T$ and define $N = \{f_n | n \in \omega_c\}$ and $f \in X^T$ such that $f_n s_m = y_{m+2}$ for $m > n$, $f_n s_m = y_{m+1}$ for $m \leq n$, $f s_m = y_{m+1}$, and $f_n[T-S] = f[T-S] = \{y_1\}$. Evidently, N does not converge $\mathcal{U}$-uniformly. $\langle N,f\rangle \in \mathcal{C}^k(\mathfrak{X}^T)$, for if all nets $\{f_n t_b | b \in B\}$ converge, then $\{t_b | b \in B\}$ must be eventually either in $T-S$ or in some $\{s_m\}$.

Corollary 1. Let $\mathfrak{X}$ be sequentially compact, $\mathcal{U}$ a uniformity inducing $\mathfrak{X}$, and N a sequence ranging in X^T. Then the following are equivalent:

(a) $\langle N,f\rangle \in \mathcal{C}^k(\mathfrak{X}^T)$.

(b) $\langle N,f\rangle \in \mathcal{C}^\sigma(\mathfrak{X}^T)$.

(c) N converges $\mathcal{U}$-uniformly to f.

Corollary 2. Let $\mathcal{U}$ be the fine uniformity of a topological space $\mathfrak{X}$. Then the following are equivalent:

(1) $\mathfrak{X}$ is sequentially compact.

(2) For each $f \in X^T$ and for every sequence N such that $\langle N,f\rangle \in \mathcal{C}^k(\mathfrak{X}^T)$, N converges $\mathcal{U}$-uniformly to f.

(3) For each $f \in X^T$ and for every sequence N such that $\langle N,f\rangle \in \mathcal{C}^\sigma(\mathfrak{X}^T)$, N converges $\mathcal{U}$-uniformly to f.

Proofs. (b)$\Rightarrow$(c) and (1)$\Rightarrow$(3) follow from Theorem 4, (c)$\Rightarrow$(a) from Theorem 3, (2)$\Rightarrow$(1) from Theorem 5, and the remaining from $\mathcal{C}^k \subset \mathcal{C}^\sigma$.

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