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TWO - NORM ALGEBRAS

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A two-norm space ([1] ,[4] ,[5]) is a triplet $(X, \| \|, \tau)$ in which X is a vector space, $\| \|$ - a norm on X, and τ a locally convex metrizable topology on X, coarser that the $\| \|$-norm topology. Therefore τ may be determined by a sequence (s_n) of seminorms. A sequence (x_n) of elements of X is called γ-convergent to x_o (in symbols $x_n \xrightarrow{\gamma} x_o$) if $\sup_n \|x_n\| < \infty$ and $\lim_{n\to\infty} s_k(x_n - x_o) = 0$ for $k = 1,2,\ldots$ All continutity concepts in such spaces will be meant in sequential sense, referred to the γ- convergence; let us call this continuity the γ-continuity. Therefore two two-norm spaces with the same carrier X are called equivalent if the resulting γ-convergence is the same in both. Since τ is coarser that the $\| \|$-norm topology, there exist constants a_n such that $s_n(x) \leq a_n \|x\|$ for each $x \in X$. Therefore $(X, \| \|, \tau)$ is equivalent to $(X, \| \|, \tau^o)$ where τ^o is the topology of the norm $\|x\|^o = \sup_n (na_n)^{-1} s_n(x)$. In this case $\|x\|^o \leq \|x\|$, and the space $(X, \| \|, \tau^o)$ will be denoted by $(X, \| \|, \| \|^o)$.

For linear maps the γ-continuity is equivalent to a topological continuity with respect to the topology constructed by A. Wiweger ([7], [8]). To describe this topology denote by $\sum_{n=1}^{\infty}{}^{*} S_n$ the set $\bigcup_{n=1}^{\infty} \sum_{k=1}^{n} S_k$, let us also denote by B the unit ball in X. The neighbourhood basis of the topology $\tilde{\tau}$ of Wiweger consists of all sets of form $\sum_{n=1}^{\infty}{}^{*} U_n \cap B$ where U_n are taken arbitrarily from a fixed neighbourhood basis $\mathcal{B}(\tau)$ of τ. Wiweger has proved that $\tilde{\tau}$ is the unique vector topology on X satisfying the conditions

(a) $\tilde{\tau}$ coincides on B with τ,

(b) any linear map from X to a locally convex topological vector space is continuous if and only if its restriction to B is continuous for the topology induced on B by τ.

The sets bounded for $\tilde{\tau}$ are precisely those which are absorbed by B. Therefore

(c) a sequence (x_n) converges γ to x_o if and only if it converges to x_o for the $\tilde{\tau}$-topology.

We shall report about some class of two-norm spaces which also are linear algebras, and for which the multiplication is γ-continuous in both variables jointly. We shall suppose without further reference that the algebras we deal with are commutative.

So let $(X, \| \|, \tau)$ be a two-norm space and an algebra; the following theorem characterizes the case when $(X, \tilde{\tau})$ is a linear topological algebra.

Theorem 1. The multiplication is continuous in both variables jointly for the topology $\tilde{\tau}$ if and only if the following conditions are satisfied

(c_1) the set $B \cdot B$ is absorbed by B.

(c_2) given any $U \in \mathcal{B}(\tau)$ there exists a $V \in \mathcal{B}(\tau)$ such that

$$(V \cap B)\, B \subset U .$$

It follows from (c_1) that the norm $\| \|$ may be replaced by an equivalent submultiplicative norm, and this leaves the convergence μ unchanged. Therefore by a two norm algebra we shall denote a two-norm space $(X, \| \|, \tau)$ which is also a linear algebra such that the multiplication is continuous for the Wiweger topology $\tilde{\tau}$. Without loss of generality we can require the norm $\| \|$ to be submultiplicative. Obviously, in two-norm algebras $x_n \xrightarrow{\mu} x_o$, $y_n \xrightarrow{\mu} y_o$ implies $x_n y_n \xrightarrow{\mu} x_o y_o$.

Let us now suppose that X admits a unit $\mathbf{1}$, let $G(X)$ denote the group of invertible elements. The inverse will be called to be μ-continuous if the following conditions are satisfied

(d_1) if $x_n \xrightarrow{\mu} x_o \in G(X)$, then almost all x_n are in $G(X)$,

(d_2) if $x_n \xrightarrow{\mu} x_o$, $x_n, x_o \in G(X)$, then $x_n^{-1} \xrightarrow{\mu} x_o^{-1}$.

The condition (d_2) is equivalent to

(d_2') if $x_n \xrightarrow{\mu} \mathbf{1}$, $x_n \in G(X)$, then $\sup_n \| x_n^{-1} \| < \infty$.

Theorem 2. The inverse in a two-norm algebra is μ-continuous if and only if $G(X)$ is open for the topology $\tilde{\tau}$ and the map $x \mapsto x^{-1}$ is continuous on $G(X)$ equipped with the topology $\tilde{\tau}$.

By a theorem of Turpin it follows

Theorem 3. Let $(X, \| \|, \tau)$ be a two-norm algebra with μ-continuous inverse, then $(X, \tilde{\tau})$ is locally m-convex.

A two norm space $(X, \| \|, \tau)$ is called non-trivial if the topology τ is not identical with the $\| \|$-norm topology. There exist non-trivial two-norm spaces for which the conditions (d_1) and (d_2)

are satisfied. Such are, for instance, [3] two-norm algebras $(X, \| \|, \| \|^o)$ in which

$$\|xy\| \leq \|x\|^o \|y\| + \|y\|^o \|x\| .$$

As an example may serve the space V of continuous functions of finite variation in an interval, with pointwise multiplication and with norms

$$\|x\| = |x(a)| + \operatorname{var}\{x(t) : a \leq t \leq b\} ,$$
$$\|x\|^o = \sup\{|x(t)| : a \leq t \leq b\} .$$

On the other hand there exists an ample class of two-norm algebras in which the inverse is not μ-continuous. Namely we have [3]

Theorem 4. Let $(X, \| \|, \| \|^o)$ be a non-trivial two-norm algebra, let $(X, \| \|)$ be a function algebra, then the condition (d_2) is not satisfied.

In algebras without unit we usually replace the inverse by the quasiinverse x^o, and $G(X)$ by the set $Q(X)$ of quasi invertible elements. A result similar to Theorem 3 holds true: if the condition (d_1) with $G(X)$ replaced by $Q(X)$ holds true and if $x_n \xrightarrow{\mu} x_o$, $x_n, x_o \in Q(X)$ implies $x_n^o \xrightarrow{\mu} x_o^o$, then the algebra $(X, \tilde{\tau})$ is m-convex.

In two-norm algebra three sets of continuous characters need to be considered: $\mathfrak{M}^o$, $\mathfrak{M}^\mu$, and $\mathfrak{M}$ composed of charaters which are continuous for the topology τ, or μ-continuous, or continuous for the $\| \|$-norm topology, respectively. Even when the algebra X has a unit, $\mathfrak{M}^o$ can be empty. If in two-norm space linear functionals, continuous for the topology τ coincide with these which are μ-continuous, then [4] the two-norm space is trivial. In contrast, in non-trivial two-norm algebras the case $\mathfrak{M}^o = \mathfrak{M}^\mu \neq \phi$ can occur.

When X has the unit and a sequence (s_n) of submultiplicative seminorms determining the topology τ exists, then $\mathfrak{M}^o \neq \phi$. In this case the set of maximal ideals closed for the topology $\tilde{\tau}$ is the union of (non-empty) sets M^k of maximal ideals closed with respect to the seminorm s_k. If we endowe the set $M^\mu = \bigcup_{k=1}^{\infty} M^k$ with the topology induced by the Gelfand topology of the set of all maximal ideals then M^k become compact. Restricting the Gelfand representation $m \mapsto \hat{x}(m)$ to the domain M^μ we obtain a representation of a two-norm algebra with submultiplicative seminorms s_n into an algebra $C(S)$ of bounded, continuous functions defined on a completely regular Hausdorff space S, such that $S = \bigcup_{n=1}^{\infty} S_n$, S_n being compact subsets of S. Setting for $u \in C(S)$

$$\|u\|_S = \sup\{|u(s)| : s \in S\},$$
$$[u]_n = \sup\{|u(s)| : s \in S_n\},$$

we obtain thus a representation $H : X \to C(S)$ satisfying the conditions

$$\|H(x)\|_S \le \|x\|,$$
$$[H(x)]_n \le s_n(x).$$

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