LOCAL CONNECTEDNESS AND FIXED POINTS

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It is well known that each plane locally connected continuum which does not separate the plane is an absolute retract and therefore has the fixed point property. In 1934, K. Borsuk described an example of a locally connected acyclic continuum in E^3 which admits a fixed point free mapping / see [1] /. On the other hand each plane locally connected continuum which separates the plane, contains a simple closed curve as its retract, and therefore admits a fixed point free mapping. K. Borsuk asked the question, whether every locally connected continuum contained in E^3 and separating E^3, must admit a fixed point free mapping / see[4] , Problem 7, p. 68 /. The aim of this note is to sketch an example, which gives the negative answer to this question.

Let J be an interval lying in E^3. By a cannal we mean a set homeomorphic to the Cartesian product of a circle S^1 and the closed half real line. Let C_1 and C_2 be two disjoint cannals converging to J as indicated in the Figure. Observe that C_1 converges to the whole of J, and both C_1 and C_2 become thinner and thinner, as we go along them. Denote by A_1 and A_2 the circles which bound C_1 and C_2, respectively. One can assume that there is a closed ball B such that the union of C_1 and C_2 is contained in the interior of B, except the circles A_1 and A_2, which are contained in the boundary of B.

Let X denote the continuum remaining after removing from B the interiors of the tunnels determined by C_1 and C_2. Observe that X is a locally connected acyclic continuum. To complete the construction of the promised example, it suffices to add to X the disc, say D, contained in the boundary of B, bounded by A_1 and missing A_2.

Denote $Y = X \cup D$. Note that Y is a locally connected continuum which separates E^3. In [3] it is proven that a two dimenisonal version of such an example has the fixed point property. The proof that Y has the fixed point property is almost the same. We shall give a brief idea of this proof.

The necessary condition for Y to have the fixed point property is noncontractibility of A_1 in X. In fact, if A_1 were contractible in X, then the identity on A_1 would have an extension to a mapping from the disc D into X. Thus there would exist a mapping f from Y to itself such that f restricted to A_1 would be a rotation, the image of D would lie in X and X would be mapped onto D. This mapping would have no fixed points.

It follows from the construction that A_1 is not contractible in X. Moreover, it is true that each continuous mapping from the two-dimensional sphere S^2 into Y induces the trivial morphism on the / Čech or Vietoris / homology groups.

The proof of this uses the fact that C_1 and C_2 are linked.

In [5] , K.Sieklucki introduced the notion of a deformation quasi-retract and proved that a deformation quasi-retract of an AR-space has the fixed point property. One can prove that a deformation quasi-retract of an ANR-space satisfies the Lefschetz fixed point theorem / see [2] /. It follows from the construction that Y is a deformation quasi-retract of an ANR-space, thus Y satisfies the Lefschetz fixed point theorem.

Let f be an arbitrary continuous mapping of Y into itself. If the image of J is contained in J we are done. Therefore there is a point t belonging to J such that $f(t)$ does not belong to J . Observe that Y is locally contractible at each point of $Y-J$. Thus f can be extended onto a disc D_1 filling up the hole determined by C_1 and lying sufficiently close to t . Observe that the union of D, D_1 and the part of C_1 cut off by D_1 forms the topological sphere. We know that a mapping from a sphere to Y is homologicaly trivial. Since the selection of the disc D_1 is free, such spheres aproximate C_1 and therefore f is also homologicaly trivial. Now, by the Lefschetz fixed point theorem for Y, we infer that there is a fixed point of the mapping f . /For more precise proof see [3] ./

R e f e r e n c e s

[1] K.Borsuk: Sur un continuum acyclique qui se laise transformer topologiquement en lui-même sans points invariants. Fund.Math. 24 (1934), 51-58.

[2] P.Minc: Generalized retracts and the Lefschetz fixed point theorem. Bull. Acad. Polon. Sci. Sér.Sci.Math.Astronom.Phys.(in press).

[3] P.Minc: Fixed points and locally connected cyclic continua in E^3;

Fund.Math. (in press).

[4] M.Rudin: Lectures on set theoretic topology.Regional Conference Series in Mathematics 23, 1975.

[5] K.Sieklucki: Generalized retractions and fixed points. Bull.Acad. Polon.Sci.Sér.Sci.Math.Astronom.Phys. 19 (1971), 945-949.

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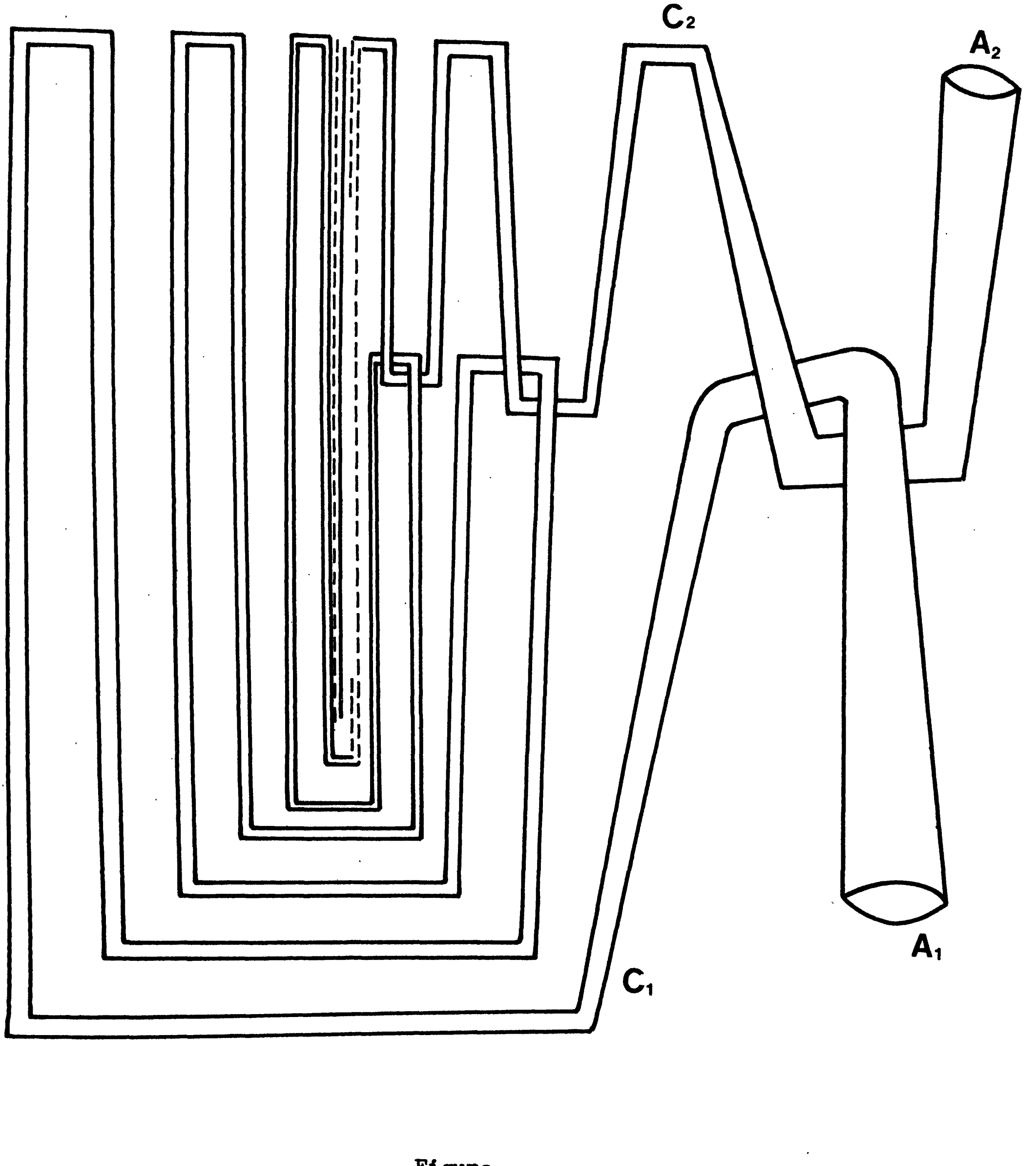

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