Marek Wilhelm

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ON CLOSED GRAPH THEOREMS

M. WILHELM

Wrocław

Let T be a topological space, let (X,d) be a complete metric space, and let f be a function on T to X. Put $df(u,v) = d(f(u),f(v))$ for $u,v \in T$; df is a pseudo-metric for T. The letter U will stand for open sets in T.

Definition. $d_f(u,v) = \sup_{U \ni u} \inf_{u' \in U} df(u',v)$, $u,v \in T$.

Theorem 1. The function d_f on $T \times T$ to R^+ has the following properties:

(i) $d_f(t,t) = 0$.

(ii) $d_f(u,v) = \inf \left\{ \sup_{\sigma} df(u_\sigma,v) \ : \ u \in \lim_{\sigma} u_\sigma \right\}$.

(iii) $d_f(u,v) \leq df(u,v) \leq d_f(u,v) + f_d(u)$, where $f_d(u) = \inf_{U \ni u} \sup_{u' \in U} df(u',u)$.

(iv) If f is continuous at t, then d_f is continuous at (t,t) and $d_f(t,v) = df(t,v)$ for all $v \in T$.

(v) $|d_f(t,u) - d_f(t,v)| \leq df(u,v)$.

(vi) d_f is lower semicontinuous in first variable.

(vii) If d_f is symmetric, then it is a pseudo-metric.

We say that f is nearly continuous at t if for any open set Y containing f(t), t is in the interior of the closure of $f^{-1}(Y)$ (cf. Kelley & Namioka [3]). If f is continuous at t, then f is nearly continuous at t.

Theorem 2. The function f is nearly continuous at t if and only if the function d_f is continuous in first variable at (t,t).

Theorem 3. (cf. [6], [4], [1]) Suppose that at least one of the following three conditions is satisfied:

(a) T is metrically topologically complete,

(b) the graph of f is metrically topologically complete in its relative product topology,

(c) the counter image of any compact set is compact.

Then the following three conditions are equivalent:

(i) f is continuous;

(ii) the graph of f is closed and f is nearly continuous;

(iii) the graph of f is closed and d_f is continuous in first variable at every point of the diagonal $\Delta(T)$.

Our central result, Theorem 4, shows that the dual statement - concerning the continuity of d_f in <u>second</u> variable - is also true. Neither of them implies the other. Notice that in Theorem 4 no assumptions like (a), (b), or (c) of Theorem 3 are necessary.
We give here a self-contained proof; another one, based on the induction theorem of Pták [5], is contained in a more extensive paper on the subject submitted to Fundamenta Mathematicae.
Let us say that the graph of f (denoted by G(f)) is closed at t if for any point $x \in X$, $(t,x) \in \overline{G(f)}$ implies $(t,x) \in G(f)$. If f is continuous at t, then the graph of f is closed at t.

Theorem 4. Let $t \in T$. The function f is continuous at t if (and only if) the graph of f is closed at t and d_f is continuous in second variable at the point $(t,t) \in \Delta(T)$.

Proof. Let $\varepsilon > 0$. Since d_f is continuous in second variable at (t,t), there are open sets U_n containing t such that

$$\forall_{u \in U_n} \; d_f(t,u) < \varepsilon 2^{-n-3},$$

that is

$$\forall_{u \in U_n} \; \forall_{U \ni t} \; \exists_{t' \in U} \; df(t',u) < \varepsilon 2^{-n-3}.$$

Choose any $v \in U_1$; it is sufficient to prove that $df(v,t) \leq \varepsilon$.
Since $v \in U_1$, there are $t^1_U \in U$ with $df(t^1_U, v) < \varepsilon 2^{-4}$.
Since $t^1_{U_2} \in U_2$, there are $t^2_U \in U$ with $df(t^2_U, t^1_{U_2}) < \varepsilon 2^{-5}$.
Continuing this process we obtain some elements $t^n_U \in U$ (where open $U \ni t$ and $n \in N$) with $df(t^{n+1}_U, t^n_{U_{n+1}}) < \varepsilon 2^{-n-4}$. The product net $\{t^n_U\}$ ($t^n_U \leq t^{n'}_{U'}$ iff $U \supseteq U'$ and $n \leq n'$) is convergent to t and

$$df(t^{n+1}_{U'}, t^n_U) \leq df(t^{n+1}_{U'}, t^n_{U_{n+1}}) + df(t^n_{U_{n+1}}, t^{n-1}_{U_n}) + df(t^n_U, t^{n-1}_{U_n}) <$$

$$\varepsilon 2^{-n-4} + \varepsilon 2^{-n-3} + \varepsilon 2^{-n-3} < \varepsilon 2^{-n-1}.$$

Hence $\{f(t^n_U)\}$ is a Cauchy net and

$$df(t_U^n,v) \leq df(t_U^n,t_U^1) + df(t_U^1,v) < \varepsilon 2^{-1} + \varepsilon 2^{-4} < \varepsilon.$$

Since the metric space (X,d) is complete and the graph of f is closed at t, the net $\{f(t_U^n)\}$ converges to $f(t)$, which implies

$$df(t,v) = \lim df(t_U^n,v) \leq \varepsilon.$$

From now on we assume that T is a topological group, (X,d) is a complete metric group with d left-invariant and f is a homomorphism on T to X.

Theorem 5. The function d_f is a left-invariant pseudo-metric for T and

$$d_f(u,v) = \sup_{\substack{U \ni u \\ V \ni v}} \inf_{\substack{u' \in U \\ v' \in V}} df(u',v') \text{ for } u,v \in T.$$

Theorem 4 together with Theorems 2 and 5 yields immediately the following result of Kelley ([2], Problem R on p.213).

Theorem 6. The homomorphism f is continuous if and only if the graph of f is closed and f is nearly continuous.

Finally, let us recall some assumptions under which the homomorphism f is automatically nearly continuous:

(1) T is of the second category and $f(T)$ is separable (cf. Weston [6], Theorem 3 on p.345),

(2) T is of the second category and T and X are linear topological spaces over the field of rationals (cf. ibidem),

(3) T and X are locally convex spaces , T is barreled and f is linear (cf. Kelley & Namioka [3], Problem E on p.106).

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Institute of Mathematics, Technical University, Wrocław, Poland