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ON A FACTORIZATION LEMMA AND A CONSTRUCTION OF ABSOLUTE WITHOUT SEPARATION AXIOMS

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The aim of this paper is to show a generalization of the construction of absolute presented in our papers [2] , [3] and [4] . We shall prove the following Factorization Lemma : each skeletal map $f : E \xrightarrow{onto} X$, where E is extremally disconnected, is a composition $E \xrightarrow{g} Z \xrightarrow{h} X$, where Z is extremally disconnected and the factor h is irreducible and separated. By the use of this Lemma we give a general construction of the absolute of a space assuming no separation axioms. We shall also prove that projective objects in the category of H-closed spaces and their continuous maps are exactly those which are either finite spaces or Katětov extensions of discrete spaces.

The concept of the absolute in the compact Hausdorff case is due to Gleason [8] (see also Rainwater [15] and Hager [9]). In more general cases several constructions were given : Iliadis [11] , Ponomarev [14] , Flachsmeyer [6] , Mioduszewski [12] , Mioduszewski and Rudolf [13] , Banaschewski [1] , Dyckhoff [5] and in the case of T_0-spaces our papers [2] , [3] and [4] . Recently general constructions of absolute also in the absence of separation axioms was given by Ul'janov [17] and Šapiro [16] . Construction of the absolute presented in [17] is a generalization of a construction of Ponomarev [14] . In [16] there is improved the method of centered families used by Iliadis [11] .

All maps are assumed to be continuous. A map $f : X \xrightarrow{onto} Y$ is irreducible if $\mathrm{cl}f(F) \neq Y$ whenever F is closed and $F \neq X$. A map is separated if distinct points with the same image have disjoint neighbourhoods. A map is perfect if it is closed and preimages of points are compact Hausdorff. A space is extremally disconnected (shortly, e.d.) if the closure of each open subset is open. A map $\alpha^X : \alpha X \xrightarrow{onto} X$ is called to be an absolute of a space X if it is irreducible, separated and perfect and the space αX is e.d.

Lemma 1. If a map $f : E \longrightarrow X$ is irreducible and separated, then for each $g : E \longrightarrow E$ for which $f \circ g = f$, g is the identity.

Proof of this lemma is an easy modification of 1.4. in [13]

Lemma 2. If a map $f : X \xrightarrow{onto} E$ is irreducible, separated and E is e.d., then f is one-to-one.

Proof. It is easy to check that if $x \in U$, then $f(x) \in cl(E \setminus clf(X \setminus U))$. If $f(x) = f(y)$ and $x \neq y$, then there exist disjoint open sets U and V such that $x \in U$ and $y \in V$; f being separated. Since $U \cap V = \emptyset$ and E is e.d.,

$$cl(E \setminus clf(X \setminus U)) \cap cl(E \setminus clf(X \setminus V)) = \emptyset ;$$

a contradiction.

A map $f : X \longrightarrow Y$ is said to be skeletal (see [13]) provided the preimage under f of each open and dense subset of Y is dense in X or, equivalently, if $Intclf^{-1}(U) = Intf^{-1}(clU)$ for each U being open in Y. It is known that each irreducible map is skeletal and that the class of all topological spaces and their skeletal maps forms a category.

Lemma 3. Irreducible separated maps are monomorphisms in the category of topological spaces and their skeletal maps.

Proof of this lemma is analogous to the proof of Lemma 4 in [3]

Theorem 1. If a map $f : E \xrightarrow{onto} X$ is irreducible and separated and E is e.d., then the following are equivalent :

(I) f is an absolute,

(II) for each map $g : Y \longrightarrow X$, Y being e.d., there exists a map $h : Y \longrightarrow E$ such that $f \circ h = g$,

(III) for each skeletal map $g : Y \longrightarrow X$, Y being e.d., there exists exactly one map $h : Y \longrightarrow E$ such that $f \circ h = g$.

Proof. 1.(I)$\Longrightarrow$(II). To prove this implication let us consider the pullback diagram

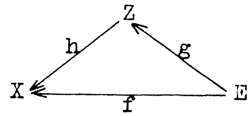
$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & T \\ g\downarrow & & \downarrow\psi \\ X & \xrightarrow[f]{} & E \end{array}$$

where $T = \{(x,y) \in Y \times E : g(x) = f(y)\}$ and φ and ψ are the restrictions of the natural projections. One can check that φ is separated and perfect. A standard method shows that there exists a closed subset Z, $Z \subset T$, such, that $\varphi|Z$ is irreducible and onto. Hence, by Lemma 2, $\varphi|Z$ is a homeomorphism. The map $\psi \circ (\varphi|Z)^{-1}$ is a desired one.

2. (II)$\Longrightarrow$(III). This implication, by Lemma 3, is obvious.

3. (III)$\Longrightarrow$(I). This implication is contained in the proof of Theorem 10 in our paper [3] .

Theorem 2 (Factorization Lemma). For each skeletal map $f : E \xrightarrow{onto} X$, where E is e.d., there exists a factorization



such that the factor $h : Z \longrightarrow X$ is irreducible and separated and Z is e.d. and the family $\{h^{-1}(U) \cap \mathrm{cl}h^{-1}(V) : U \text{ and } V \text{ are open in } X\}$ is a base of the topology in Z.

Proof. Consider an equivalence on X assuming $x \sim y$ whenever the following condition holds :

$f(x) = f(y)$ and for each U and V being open in X there is $x \in f^{-1}(U) \cap \mathrm{cl}f^{-1}(V)$ iff $y \in f^{-1}(U) \cap \mathrm{cl}f^{-1}(V)$.

In the sequel the proof does not differ from the proof of Theorem 1 in our paper [2]. We define Z to be the set of all equivalence class of the relation "$\sim$" with the topology generated by the family $\{g(f^{-1}(U) \cap \mathrm{cl}f^{-1}(V)) : U \text{ and } V \text{ are open in } X\}$, where g is the projection.

Note. The Factorization Lemma proved here differs from the Factorization Lemma from our paper [2]. Namely, the equivalence relation "$\sim$" here is finer than that from [2] (they coincide in the case when X is a T_0-space).

Construction of the absolute.

Let X be a topological space. Consider all skeletal maps $f : E \longrightarrow X$, where E is e.d. These maps do not necessarily form a set. By Theorem 2, for each such a map there exists a factorization $E \xrightarrow{g} Z \xrightarrow{h} X$, where Z is e.d. and h is irreducible and separated. Since the family $\{h^{-1}(U) \cap \mathrm{cl}h^{-1}(V) : U \text{ and } V \text{ are open in } X\}$ is a base in Z and h is separated, $|Z| \leq |X| \cdot \exp|\mathcal{T}|$, where $\mathcal{T}$ is the topology in X. Hence there exists a set $S(X)$ of irreducible and separated maps $g : Y_g \longrightarrow X$ from e.d. spaces Y_g onto X such that each skeletal map $f : Y \longrightarrow X$, where Y is e.d., admits a decomposition $Y \longrightarrow Y_g \xrightarrow{g} X$ for some $g \in S(X)$. It was proved in [3] (see Lemma 7) that for each topological space X the set $S(X)$ is non-empty; for each topological space there exists an e.d. topology which is maximal in the set of all topologies on X having skeletal contraction onto X. Let $\tilde{Y}$ be the disjoint union of all Y_g for $g \in S(X)$ and let $\tilde{f} : \tilde{Y} \xrightarrow{\text{onto}} X$ be the map induced by maps from $S(X)$. Clearly, $\tilde{Y}$ is e.d. and $\tilde{f}$ is skeletal ; $\tilde{f}|Y_g$ being skeletal (irreducible maps are skeletal). By Theorem 2, there exists a factorization $\tilde{Y} \longrightarrow \alpha X \xrightarrow{\alpha^X} X$, where α^X is irreducible and separated. Moreover, for each skeletal map $f : Y \longrightarrow X$, where Y is e.d., there exists a map $h : Y \longrightarrow \alpha X$ such that $\alpha^X \circ h = f$. Hence, by Theorem 1, $\alpha^X : \alpha X \longrightarrow X$ is the absolute. By Lemma 1, for each topological space the absolute is unique up to a homeomorphism.

Remark of the categorial character.

Let us consider the category TOP of all topological spaces and their skeletal maps. By Theorem 1, for each skeletal map $f : X \longrightarrow Y$ there exists a unique map $\hat{f} : \alpha X \longrightarrow \alpha Y$ such that $\alpha^Y \circ \hat{f} = f \circ \alpha^X$, $\alpha^X : \alpha X \longrightarrow X$ and $\alpha^Y : \alpha Y \longrightarrow Y$ being the absolutes. Hence the absolute define a functor α: TOP $\longrightarrow$ ED , ED being the category of all e.d. spaces and their skeletal maps. By Factorization Lemma, our construction of the absolute falls under a general categorial scheme of constructions of adjoint functors given by Freyd [7] , dual to the construction of the Čech-Stone functor.

A map $h : E \longrightarrow \alpha X$, E being e.d., is said to be lifting over αX of a map $f : E \longrightarrow X$ provided that $f = \alpha^X \circ h$, where $\alpha^X : \alpha X \longrightarrow X$ is the absolute.

Theorem 3. A map $f : E \longrightarrow X$, where E is e.d., admits a unique lifting over αX iff

$$(*) \qquad \mathrm{Int\,cl} f^{-1}(U) = \mathrm{Int} f^{-1}(\mathrm{cl} U)$$

for each U being regularly open subset of X.

The proof of this theorem is the same as the proof of Theorem 3 in our paper [4] .

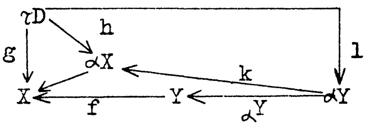
Note. The maps for which condition (*) holds were considered by Henriksen and Jerison [10] (see also [13] and[16]) and are called in the literature HJ-maps. However this class is not closed with respect to the superposition. So, from the categorial point of view the class of skeletal maps seems to be the best subclass of the class of HJ-maps.

An object P of the category K is said to be projective in K provided for each epimorphism $f : Y \longrightarrow X$ from K and for each $g : P \longrightarrow X$ from K there exists a morphism $h : P \longrightarrow X$, $h \in$ K , such that $g = f \circ h$. Gleason [8] proved that in the category of compact Hausdorff spaces and their continuous maps, projective objects coincide with e.d. spaces. We shall prove the following

Theorem 4. Projective objects in the category of H-closed spaces and their continuous maps are exactly those which are either finite spaces or Katětov extensions of discrete spaces.

Proof.1. The necessity was proved in our paper [4] (Theorem 5).
2. To prove the converse let $g : \tau D \longrightarrow X$ and let $f : Y \longrightarrow X$ be a map onto, X and Y being H-closed, τD denote the Katětov extension of

discrete space D. Consider the following diagram



The topologies of spaces τD, αX, αY have contractions to compact Hausdorff ones (such topologies are called compact-like), being H-closed and e.d. The maps h and k exist in virtue of Theorem 1. Since $f \circ \alpha^Y$ is onto and α^X is irreducible and αY is H-closed, k is onto. Thus, by Theorem 6 in our paper [4] , there exists a map $l : \tau D \longrightarrow \alpha Y$ such that $k \circ l = h$. Therefore $g = f \circ \alpha^Y \circ l$. The map $\alpha^Y \circ l$ is the desired one.

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