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PLANABLE AND SMOOTH DENDROIDS

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§ 1. Introduction. All spaces considered in this paper are metric and compact. A continuum means a compact, connected space. A dendroid is a hereditarily unicoherent and arcwise connected continuum. If a dendroid has only one ramification point t (see [3], p. 230), it is called a fan with the top t (see [5], p. 6). A unique arc joining points a and b in a given dendroid X we denote by ab. A dendroid X is said to be smooth at p provided $\lim a_n = a$ implies $\mathrm{Lim}\, pa_n = pa$ (see [8], p. 298). If a dendroid X has a point p, at which it is smooth, then we say simply that X is smooth.

A space X is said to be planable if there is a homeomorphism of X into the Euclidean plane. It is well known that the problem of a characterization of continua X which are not planable is solved in case when X is locally connected. Namely, a locally connected continuum X, which is not the two-sphere, is planable if and only if it contains no homeomorphic image of the Kuratowski's primitive skew graphs K_1 and K_2 (see [14]) and of the Claytor's curves C_1 and C_2 (see [11]). The problem of the planability of continua which are not locally connected is open (for some partial results see [1], Theorem 4 and Example 1, p. 654). Even for dendroids this problem is very complicated. There is no finite (countable) collection $\mathcal{A}$ of dendroids such that any not planable dendroid (smooth dendroid) contains a homeomorphic copy of some member of $\mathcal{A}$ (see [6] and [9]). Exactly the same situation is for fans, which can be not planable (the first example was given in [2]). Namely, there does not exist also such countable collection $\mathcal{A}$ for fans (see [16]). All smooth fans are planable, because they can be imbedded in the Cantor fan (see [5], Theorem 9, p. 27 and [12], Corollary 4, p. 90).

Recall that if A is a closed subset of a space X, then the point $a \in A$ is called an inaccessible point of A in X provided there is no non-degenerate arc ab in X such that $ab \cap A = \{a\}$.

Some sufficient conditions to the non-planability of dendroids in terms of inaccessible points are proved in [10].

A continuum K_0 is said to be a convergence continuum of X if it is the topological limit of a sequence of continua K_n such that $K_0 = \mathrm{Lim}\, K_n$ and $K_n \cap K_m = \emptyset$ for $n \neq m$ and $n,m = 0,1,2,\ldots$ (see [15], p. 245).

It is easy to see (for example from Claytor's result) that every locally connected dendroid is planable. Thus, since the non-local connectedness of a given continuum X implies the existence of non-degenerate convergence subcontinua of X (see [15], § 49, VI, Theorem 1, p. 245) it seems possible to characterize planable dendroids in terms of the convergence continua and of inaccessible points. In this paper we prove some results in this direction of investigation of nonplanable dendroids.

§ 2. Convergence continua of arcs. In this section we will prove that any convergence subcontinuum of planable dendroid is a convergence continuum of arcs. Firstly, from Brouwer's reduction theorem easily we obtain the following

THEOREM 1. Let a sequence $\{K_n\}$ of subcontinua of X be such that $\mathrm{Lim}\, K_n = K_0$, $K_n \cap K_m = \emptyset$ for $n \neq m$ and $n,m = 0,1,2,\ldots$ Then there is a maximal subcontinuum Q_0 of K_0 for which there are arcs $a_{n_i} b_{n_i}$ converging to Q_0 such that $a_{n_i} b_{n_i} \subset K_{n_i}$ for some subsequence $\{n_i\}$ of the sequence of natural numbers.

The following theorem generalizes Proposition 8 from [10].

THEOREM 2. Let a dendroid X contain a sequence of mutually disjoint simple triods $T_n = a_n^1 p_n \cup a_n^2 p_n \cup a_n^3 p_n$ $(n = 1,2,\ldots)$, where a_n^1, a_n^2, a_n^3 are endpoints and p_n is the top of T_n, and such that $A^i = \mathrm{Lim}\, a_n^i p_n$, $T_n \cap \bigcup_{i=1}^{3} A_i = \emptyset$ and $b^i \in A^i \setminus \bigcup_{j \neq i} A^j$ for $i,j = 1,2,3$ and $n = 1,2,\ldots$ Then X is not planable.

P r o o f. Suppose X can be imbedded in the plane R^2 under a homeomorphism $h: X \to h(X) \subset R^2$. We will write x instead of $h(x)$ to simplify denotations. Let B^i be regions in R^2 such that $b^i \in B^i$ and $\overline{B}^i \cap \left(\bigcup_{j \neq i} (A^j \cup \overline{B}^j) \right) = \emptyset$ for $i,j = 1,2,3$. Thus, since $b^i \in A^i = \mathrm{Lim}\, a_n^i p_n$, we can assume that

(1) $a_n^i p_n \cap B^i \neq \emptyset$, $a_n^i p_n \cap \left(\bigcup_{j \neq i} \overline{B}^j \right) = \emptyset$

for $i,j = 1,2,3$ and for each $n = 1,2,\ldots$

The arc $a_1^1 p_1 \cup a_1^2 p_1$ contains an arc $c^1 c^2$ such that $c^1 c^2 \cap \left(\bigcup_{i=1}^{3} (A^i \cup \overline{B}^i) \right) = c^1 c^2 \cap (\overline{B}^1 \cup \overline{B}^2) = \{c^1, c^2\}$. Similarly, the arc $a_1^2 p_1 \cup a_1^3 p_1$ contains an arc $d^2 d^3$ such that $d^2 d^3 \cap \left(\bigcup_{i=1}^{3} (A^i \cup \overline{B}^i) \right) = d^2 d^3 \cap (\overline{B}^2 \cup \overline{B}^3) = \{d^2, d^3\}$.

Let pa^i be an arc in A^i such that $pa^i \cap \overline{B}^i = \{a^i\}$, and $pa^i \cap pa^j = \{p\}$ for $i \neq j$ and $i,j = 1,2,3$. Then the continuum $\bigcup_{i=1}^{3} (pa^i \cup \overline{B}^i) \cup c^1 c^2 \cup d^2 d^3$ separates the plane into three regions D^1, D^2 and D^3 such that $pa^i \setminus \overline{D}^i \neq \emptyset$ for $i = 1,2,3$. Infinitely many points p_n belong to D^i for some $i = 1,2,3$. Let $p_{n_j} \in D^i$ for $j = 1,2,\ldots$ It follows from (1) that $a_{n_j}^i p_{n_j} \subset D^i$. Therefore $A^i \subset \overline{D}^i$, because $A^i = \mathrm{Lim}\, a_n^i p_n = \mathrm{Lim}\, a_{n_j}^i p_{n_j}$. But $pa^i \subset A^i$ and $pa^i \setminus D^i \neq \emptyset$, a contradiction.

Now we will prove

THEOREM 3. Let a sequence of subcontinua $\{K_n\}$ of planable dendroid X be such that $\mathrm{Lim}\, K_n = K_0$ and $K_n \cap K_m = \emptyset$ for $n \neq m$ and $n,m = 0,1,2,\ldots$ Then there is a sequence $\{a_{n_i}^1 a_{n_i}^2\}$ of arcs such that $\mathrm{Lim}\, a_{n_i}^1 a_{n_i}^2 = K_0$ and

$a^1_{n_i}a^2_{n_i} \subset K_{n_i}$ for $i = 1,2,\dots$

P r o o f. Let Q_o be a maximal subcontinuum of K_o for which there are arcs $a^1_{n_i}a^2_{n_i}$ converging to Q_o and such that $a^1_{n_i}a^2_{n_i} \subset K_{n_i}$ for some subsequence $\{n_i\}$ of the sequence of natural numbers (such Q_o exists by Theorem 1). Suppose, on the contrary, that $K_o \setminus Q_o \neq \emptyset$. Let $a^3 \in K_o \setminus Q_o$. Since $K_o = \operatorname{Lim} K_n = \operatorname{Lim} K_{n_i}$, we infer that there are points $a^3_{n_i}$ belonging to K_{n_i} such that $\lim a_{n_i} = a^3$. For each $i = 1,2,\dots$ we take an arc $a^3_{n_i}p_{n_i}$ in K_{n_i} such that $a^3_{n_i}p_{n_i} \cap a^1_{n_i}a^2_{n_i} = \{p_{n_i}\}$. Since X is compact, we can assume that sequences $\{a^j_{n_i}p_{n_i}\}$ are convergent for $j = 1,2,3$. Put $A^j = \operatorname{Lim} a^j_{n_i}p_{n_i}$ for $j = 1,2,3$. By the choice of Q_o we conclude that there is a natural number i_o such that for each $i > i_o$ the set $T_{n_i} = a^1_{n_i}p_{n_i} \cup a^2_{n_i}p_{n_i} \cup a^3_{n_i}p_{n_i}$ is a simple triod. Moreover,

(2) $\quad A^j \setminus \left(\bigcup_{k\neq j} A^k\right) \neq \emptyset$ for $j,k = 1,2,3$.

In fact, suppose that $A^j \setminus \left(\bigcup_{k\neq j} A^k\right) = \emptyset$ for some $j = 1,2,3$. Then $A^j \subset \bigcup_{k\neq j} A^k$. Thus $\operatorname{Lim}\left(\bigcup_{k\neq j} a^k_{n_i}p_{n_i}\right) = \operatorname{Lim}\left(\bigcup_{k=1}^{3} a^k_{n_i}p_{n_i}\right) = A^1 \cup A^2 \cup A^3$. But sets $\bigcup_{k\neq j} a^k_{n_i}p_{n_i}$ are arcs for $i > i_o$ and Q_o is a proper subcontinuum of $A^1 \cup A^2 \cup A^3$, because $a^3 \in (A^1 \cup A^2 \cup A^3) \setminus Q_o$. It is impossible, by the choice of Q_o. The condition (2) and Theorem 2 imply that X is not planable, a contradiction. The proof of Theorem 3 is complete.

From Theorem 3 we infer that

COROLLARY 4. Any convergence subcontinuum of planable dendroid X is a convergence continuum of arcs which are contained in X.

§ 3. Some properties of planable dendroids. We have (see [3], (47), p. 239, [4], XI, p. 217 and [15], §49, III, Theorem 10, p. 470)

PROPOSITION 5. If X is a plane dendroid, then the set $R^2 \setminus X$ is connected.

Firstly, we will show the following

LEMMA 6. If $a_1,\dots,a_n$ are different accessible points of a continuum A in a plane dendroid X, then there are nondegenerate mutually disjoint arcs $a_1b_1,\dots,a_nb_n$ in X and a simple closed curve C in R^2 such that $a_ib_i \cap A = \{a_i\}$ for $i = 1,2,\dots,n$ and

$$\left(A \cup \bigcup_{i=1}^{n} a_ib_i\right) \cap C = \{b_1,b_2,\dots,b_n\}.$$

In fact, since points $a_1,\dots,a_n$ are different and since they are accessible points of a continuum A in a dendroid X, we infer that there are nondegenerate mutually disjoint arcs $a_1c_1,\dots,a_nc_n$ in X such that $a_ic_i \cap A = \{a_i\}$ for $i = 1,2,\dots,n$. Sets A and $B = \{c_1,\dots,c_n\}$ are disjoint and closed sets which do not separate the plane (cf. Proposition 5) Thus, by Theorem 9 in [15], § 61, II, p. 514, we obtain that there is a simple closed curve C in R^2 which separates A and B. Therefore $C \cap A = \emptyset$ and $C \cap a_ic_i \neq \emptyset$ for each $i = 1,2,\dots,n$. Let b_i be the first point in the arc a_ic_i (in the order from a_i to c_i) which belongs to C for $i = 1,2,\dots,n$. Then C and arcs $a_1b_1,\dots,a_nb_n$ satisfy required conditions.

Let X be a dendroid and let A be a subcontinuum of X. A point b of A is called a <u>convergence point</u> of A in X if there is a sequence $\{a_n\}$ of X such that $\mathrm{Lim}\, a_nb = A$ and $\mathrm{Lim}\,(a_nb \cap A) = \{b\}$.

It follows from the definition of the convergence point that

LEMMA 7. If b is a convergence point of a subcontinuum A of a dendroid X, then b belongs to the closure of the set of all accessible points of A in X.

Now, we will prove

THEOREM 8. Let b be a convergence point of a subcontinuum A of a planable dendroid X. Then the set of all accessible points of A in X is contained in some arc cb.

P r o o f. We assume that X is embedded in the plane R^2. Firstly we will prove that

(3) the set of all accessible points of A in X is contained in some arc c_1c_2.

In fact, suppose, on the contrary, that c_1, c_2 and c_3 are accessible points of A in X and they are endpoints of a simple triod T contained in A. According to Lemma 6 there are disjoint nondegenerate arcs d_1c_1, d_2c_2 and d_3c_3 in X and a simple closed curve C in R^2 such that

(4) $d_ic_i \cap A = \{c_i\}$ for $i = 1,2,3$ and $\left(A \cup \bigcup_{i=1}^{3} d_ic_i\right) \cap C = \{d_1, d_2, d_3\}$.

The curve $D = C \cup \bigcup_{i=1}^{3} d_ic_i \cup T$ separates the plane R^2 into four domains such that the closure of any of them fails to contain at least one of the points c_1, c_2, c_3. Since b is a convergence point of A in X, we conclude, by (4), that there are arcs a_na_n' in X such that $\mathrm{Lim}\, a_na_n' = A$ and $a_na_n' \cap D = \emptyset$ for each $n = 1,2,\dots$ Therefore some subsequence $\{a_{n_k}a_{n_k}'\}$ of the sequence $\{a_na_n'\}$ is contained in some domain into which D separates the plane. Then the set $\mathrm{Lim}\, a_{n_k}a_{n_k}'$ fails to contain some c_i. But $\{c_1, c_2, c_3\} \subset T \subset A = \mathrm{Lim}\, a_na_n' = \mathrm{Lim}\, a_{n_k}a_{n_k}'$, a contradiction.

From Lemma 7, we infer that $b \in c_1c_2$. Suppose, on the contrary, that

r_1 and r_2 are accessible points of A in X such that $c_1 \leq r_1 < b < r_2 \leq c_2$ (in the natural order of c_1c_2). According to Lemma 6 there are disjoint nondegenerate arcs s_1r_1 and s_2r_2, and a simple closed curve S in R^2 such that

(5) $s_ir_i \cap A = \{r_i\}$ for $i = 1,2$ and $(A \cup s_1r_1 \cup s_2r_2) \cap S = \{s_1, s_2\}$.

The curve $R = S \cup s_1s_2$ separates the plane R^2 into three domains W_1, W_2 and W_3. Since b is a convergence point of A in X, we infer that there is a sequence $\{a_n\}$ of points of X such that

(6) $\operatorname{Lim} a_nb = A$,

and

(7) $\operatorname{Lim} (a_nb \cap A) = \{b\}$.

We may assume, by (6), that $a_nb \cap S = \emptyset$ for each $n = 1,2,\ldots$, because $A \cap S = \emptyset$ by (5). Moreover, since sets $a_nb \cap s_1s_2$ are connected for each $n = 1,2,\ldots$, we may assume that all arcs a_nb are contained in the closure of one of sets W_1, W_2 and W_3. Say

(8) $a_nb \subset \overline{W}_1$ for each $n = 1,2,\ldots$

From (6) and (7), we infer that there is a nondegenerate arc r_3s_3 in $X \cap W_1$ such that $r_3s_3 \cap (A \cup s_1r_1 \cup s_2r_2) = r_3s_3 \cap A = \{r_3\}$. Thus r_3 is an accessible point of A in X. Therefore, by (3), we conclude that $r_1 < r_3 < r_2$ (in the order of the arc r_1r_2).

Sets A and $B = \{s_1, s_2, s_3\}$ are disjoint and closed, and they do not separate the plane (cf. Proposition 5). We obtain that there is a simple closed curve S' in R^2 which separates A and B (see [15], § 61, II, Theorem 9, p. 514). Therefore $S' \cap A = \emptyset$ and $S' \cap r_is_i \neq \emptyset$ for $i = 1,2,3$. Let s_3' be the first point in the arc r_3s_3 (in the order from r_i to s_i) which belongs to S', and let $[s_1's_2']$ be an arc in S' containing s_3' such that $[s_1's_2'] \cap (S \cup s_1s_2) = \{s_1', s_2'\}$. Then the set $[s_1's_2'] \cup r_3s_3'$ separates W_1 into three components V_1, V_2, V_3, the closure of each of them does not contain both r_1 and r_2.

Since $[s_1's_2'] \cap a_nb \subset S' \cap a_nb$ for each $n = 1,2,\ldots$ and $S' \cap A = \emptyset$, we can assume, by (6), that $[s_1's_2'] \cap a_nb = \emptyset$ for each $n = 1,2,\ldots$ Therefore, because sets $a_nb \cap (s_1s_2 \cup r_3s_3)$ are connected for each $n = 1,2,\ldots$, we infer from (8) that for each $n = 1,2,\ldots$ the arc a_nb is contained in the one of sets $\overline{V}_1 \cup r_3b$, $\overline{V}_2 \cup r_3b$, $\overline{V}_3 \cup r_3b$. Therefore some subsequence $\{s_{n_k}b\}$ is contained, say in $\overline{V}_1 \cup r_3b$. But set $\overline{V}_1 \cup r_3b$ does not contain either r_1 or r_2, and $\{r_1, r_2\} \subset A = \operatorname{Lim} a_{n_k}b = \operatorname{Lim} a_nb$, a contradiction. The proof of Theorem 8 is complete.

Combining Lemma 7 and Theorem 8 it is easy to obtain

COROLLARY 9. If A is a subcontinuum of planable dendroid X, then A has at most two convergence points.

Remark that if one will change the definition of convergence points distinguishing two situations, when sets $a_n b \cap A$ are degenerate and when they are nondegenerate, then he may prove other properties of planable dendroids, which do not follow from above proved properties.

§ 4. Two examples of plane smooth dendroids. It is known (see [13], Corollary 4.2) that there is no universal plane dendroid, i.e., there is no plane dendroid containing a homeomorphic copy of any plane dendroid. In spite of this one can ask whether there is a plane smooth dendroid which contains a homeomorphic copy of any plane smooth dendroid. The answer is negative. We consider firstly two special examples of plane smooth dendroids to obtain this result.

Let (x,y,z) denote a point of the Euclidean 3-space having x,y and z as its rectangular coordinates. Put

$$D_1 = \bigcup_{n=1}^{\infty} \Big(\{ (1/n \cos t, 1/n \sin t, 0) : 0 \le t \le 3/2\pi \} \cup \cup \{(x,-1/n,0) : 0 \le x \le 1\} \cup \{(x,0,0) : 0 \le x \le 1\} \Big),$$

$$D_2 = \bigcup_{n=1}^{\infty} \Big(\{(t - \tfrac{1-t}{n}, \tfrac{t}{n}, 0 : 0 \le t \le 1\} \cup \{(-t + \tfrac{1-t}{n}, -\tfrac{t}{n}, 0) : 0 \le t \le 1\} \cup \cup \{(x,0,0) : -1 \le x \le 1\} \Big),$$

$$p = (0,0,0),$$
$$I = \{(0,0,z) : 0 \le z \le 1\}$$

and

$$E_i = D_i \cup I \quad \text{for } i = 1,2.$$

It is easy to see that

PROPOSITION 10. D_1 and D_2 are both smooth plane dendroids with p as a unique point at which they are smooth.

One can prove more general

PROPOSITION 10′. If X is a smooth dendroid containing either D_1 or D_2 which is contained in the plane, then p is a unique point, at which X is smooth.

We have also

PROPOSITION 11. E_1 and E_2 are both nonplanable dendroids.

Smooth dendroids have the following property

PROPOSITION 12. If a dendroid X is smooth at r, A is a subcontinuum of X, and rq is an arc such that $rq \cap A = \{q\}$, then A is smooth at q.

Now, we will prove

THEOREM 13. There is no smooth plane dendroid containing a homeomorphic copy of D_1 and a homeomorphic copy of D_2.

P r o o f. Suppose, on the contrary, that X is plane dendroid which is smooth at r and for i = 1,2 a mapping $h_i\colon D_i \to h_i(D_i)$ is a homeomorphism such that $h_i(D_i) \subset X$. Let rq_i be an arc in X such that $rq_i \cap h_i(D_i) = \{q_i\}$ for i = 1,2. By Proposition 12 we obtain that for i = 1,2 the dendroid $h_i(D_i)$ is smooth at $h_i(q_i)$. Thus $h_i(q_i) = h_i(p)$ for i = 1,2, by Proposition 10. Therefore, for i = 1,2, if the arc rq_i is nondegenerate, then the continuum $rq_i \cup h_i(D_i)$ is homeomorphic to E_i, and, by Proposition 11, we obtain a contradiction. Hence $h_1(p) = h_2(p)$. But $h_1(p)$ is an endpoint of $h_1(D_1)$ and there are two arcs in $h_2(D_2)$ having only the point $h_2(p)$ in the common part. Thus X must contain a homemorphic copy of E_1. But this is impossible by Proposition 11, because X is planable.

COROLLARY 14. There is no universal smooth plane dendroid.

§ 5. Problems. Besides the general open problem of a characterization of plane (smooth) dendroids the following problems are open.

Does a plane dendroid exist containing all plane smooth dendroids ?

Is an open image of a plane dendroid always a plane dendroid ? (compare [7]).

Remark that open mappings do not preserve the planability in general (see [17], Example, p. 189).

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