ATOMS IN UNIFORMITIES AND PROXIMITIES

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The aim of the communication is to summarize some known results on atoms in the lattices of basic continuous structures and to announce some new results concerning complementation in the lattice of uniformities.

Consider the set of all uniformities on a given set. Then the relation $\prec$ ("is finer than") makes this set a complete lattice. The zero of the lattice is the uniformly discrete uniformity while the unit is the indiscrete one. Thus, atoms in this lattice are uniformities $\mathcal{U}$ such that $\mathcal{V} \prec \mathcal{U}$ implies $\mathcal{V} = \mathcal{U}$ or $\mathcal{V}$ is uniformly discrete. Further, a complement to a uniformity $\mathcal{U}$ is such a uniformity $\mathcal{V}$ that $\mathcal{U} \wedge \mathcal{V}$ is uniformly discrete and $\mathcal{U} \vee \mathcal{V}$ is indiscrete. Analogously for proximities and topologies.

I. ATOMS

1. Let $\mathcal{F}$ be an ultrafilter on a set X, let $x_0 \in X$. Denote $T_{\mathcal{F}}$ the topology on X such that $x_0 \in \bar{M} \Longleftrightarrow M \in \mathcal{F}$ or $x_0 \in M$ and such that other points of X are isolated. Then the following is well-known and easy.

<u>Proposition</u>. Atoms in the lattice of topologies on X are just topologies of the form $T_{\mathcal{F}}$. Each topology is a supremum of atoms.

2. Let $\mathcal{F}$, $\mathcal{G}$ be two distinct ultrafilters on X. Denote $p_{\mathcal{F}\mathcal{G}}$ the proximity on X such that two disjoint sets A, B are proximal iff $A \in \mathcal{F}$ and $B \in \mathcal{G}$ or conversely.

<u>Proposition[3]</u>. Atoms in the lattice of proximities on X are just proximities of the form $p_{\mathcal{F}\mathcal{G}}$. Each proximity is a supremum of atoms.

3. Let $\mathcal{F}$ be an ultrafilter on X and $f : X \rightarrow X$ a bijection such that $f\mathcal{F} \neq \mathcal{F}$. Denote $S_{\mathcal{F}}$ the uniformity on X a base of which consists of covers of the form $\{\{x,fx\} \mid x \in F\} \cup \{\{x\} \mid x \in X\}$,

where $F \in \mathfrak{F}$.

Theorem[1]. Proximally non-discrete atoms in the lattice of uniformities on X are just uniformities of the form $S_{\mathfrak{F}}$.

The uniformity $S_{\mathfrak{F}}$ induces the proximity $p_{\mathfrak{F} \, f \mathfrak{F}}$ and is minimal, but not necessarily the finest one with this property. In other words, $S_{\mathfrak{F}}$ need not be proximally fine (a uniformity $\mathcal{U}$ is proximally fine if any uniformity with the same proximity is coarser than $\mathcal{U}$). It appears that the question whether $S_{\mathfrak{F}}$ is proximally fine leads to examination of ultrafilters. Namely, consider the following properties of an ultrafilter $\mathfrak{F}$ on a countable set N.

(PF) $S_{\mathfrak{F}}$ is proximally fine;

(OPF) $S_{\mathfrak{F}}$ is proximally fine w.r.t. 0-dimensional uniformities;

(Sel) $\mathfrak{F}$ is selective (= minimal in the Rudin-Keisler order);

(R) If $f,g : N \longrightarrow N$ are two mappings such that $f\mathfrak{F} = g\mathfrak{F}$ then there is $F \in \mathfrak{F}$ with $f/F = g/F$;

(P) If $f,g : N \longrightarrow N$ are two one-to-finite relations (i.e., for every n, fn and gn are finite subsets of N) such that $fF \cap gF \neq \emptyset$ for every $F \in \mathfrak{F}$, then either there exists $F \in \mathfrak{F}$ such that $fn \cap gn \neq \emptyset$ for every $n \in F$ or there is $n \in N$ with $f^{-1}n \cup g^{-1}n \in \mathfrak{F}$.

Theorem[2]. (Sel) $\Longleftrightarrow$ (P) $\Longrightarrow$ (PF) $\Longrightarrow$ (OPF) $\Longleftrightarrow$ (R).

It remains open whether (PF) $\Longrightarrow$ (P) and (OPF) $\Longrightarrow$ (PF) hold. Neverthless, Alain Louveau pointed out to us that (R) fails to imply (Sel) so that the implications in question cannot hold simoultaneously.

4. Let $\mathfrak{F}$ be an ultrafilter on X. Denote $A_{\mathfrak{F}}$ the uniformity on X consisting of all covers P with $P \cap \mathfrak{F} \neq \emptyset$.

Theorem[1]. $A_{\mathfrak{F}}$ is an atom iff $\mathfrak{F}$ is selective. Each proximally discrete atom refines some $A_{\mathfrak{F}}$.

If $\mathfrak{F}$ is an ultrafilter on X and a uniformity A_x on a set Y_x is given for every $x \in X$ then all covers of $\bigcup Y_x$ of the form $\bigcup \{ P_x \mid x \in F\} \cup \{\{x\} \mid x \in \bigcup Y_x\}$, where $F \in \mathfrak{F}$ and P_x is in A_x for each $x \in F$, form a basis of a uniformity which will be denoted by $\sum_{\mathfrak{F}} A_x$. If each A_x is an atom so is $\sum_{\mathfrak{F}} A_x$. Thus, assuming the existence of selective ultrafilters, we can construct proximally discrete atoms on arbitrary cardinalities.

All atoms described above are 0-dimensional(= have a basis consisting of partitions). This is also the case of beautiful examples of atoms constructed by P. Simon (see this volume). The problem of the existence of a non-0-dimensional atom raised in[1]has been solved affirmatively very recently by the second author and V. Rödl(to appear).

II. COMPLEMENTS

Complements in the lattice of topologies have been investigated in a lot of papers; see [4]for a comprehensive list of results. It was shown that this lattice is complemented[5]. It is far from being true in case of the lattice of uniformities: if a uniformity is proximally discrete then it has no complement. Another example of a uniformity without complement is the uniformity on $X = \{1/n \; ; \; n = 1, 2, 3 \dots\}$ induced by the usual metric on the reals. No characterization of uniformities having a complement is known to us. We agreed to focus our attention to the simplest case: to the metric uniformities on a countable set. The reduction to the metric case is not too restrictive:

<u>Proposition</u>. If $\mathcal{U}$ is a complement to $\mathcal{V}$ then there exists pseudometrics $\rho \in \mathcal{U}$, $\sigma \in \mathcal{V}$ such that $\mathcal{U}_\rho$ is a complement to $\mathcal{V}_\sigma$ where $\mathcal{U}_\rho$ and $\mathcal{V}_\sigma$ denotes the uniformity induced by ρ and σ respectively.

Further, the countable case can be generalized to the separable one by means of the following

<u>Proposition</u>. Let $(X, \mathcal{U})$ be a dense subspace of $(Y, \mathcal{V})$. If $\mathcal{U}$ has a complement, so does $\mathcal{V}$.

Thus, our main theorem characterizes separable metric uniformities with complement.

<u>Theorem</u>. Let $\mathcal{U}$ be a separable metric uniformity on X. Then $\mathcal{U}$ has a complement iff at least one of the following conditions holds.

(i) $(X, \mathcal{U})$ admits two disjoint uniformly discrete subspaces which are proximal;

(ii) the subspace of $(X, \mathcal{U})$ consisting of all non-isolated points is neither finite nor uniformly homeomorphic to the subspace $\{1/n \; ; \; n = 1, 2, 3, \dots\}$ of the reals.

It is not surprising that complements of uniformities are not determined uniquely. The following two propositions show that the class of all complements to a given uniformity can be very extensive. Denote $\mathcal{D}$ the uniformity on the set N of integers > 0 defined by the metric

$$d(n,n) = 0, \quad d(2n-1, 2n) = 1/n, \quad d(n,k) = 1 \text{ otherwise.}$$

Futher , denote Q a uniformity on N such that (N, Q) is uniformly homeomorphic to the rationals.

Proposition. Let $\mathcal{U}$ be a metric uniformity on N without isolated points. Then there exists a complement $\mathcal{V}$ to $\mathcal{D}$ such that $(N, \mathcal{V})$ is uniformly homeomorphic to $(N, \mathcal{U})$.

Proposition. Let $\mathcal{U}$ be a uniformity on a countable set X such that $(X, \mathcal{U})$ contains a subspace which is a copy of $(N, \mathcal{D})$. Then there exists a complement $\mathcal{V}$ to Q such that $(N, \mathcal{V})$ is uniformly homeomorphic to $(X, \mathcal{U})$.

References

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