Jean Schmets

On spaces of vector-valued continuous functions

In: Josef Novák (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part B: Contributed Papers. Society of Czechoslovak Mathematicians and Physicist, Praha, 1977. pp. [400]--406.

Persistent URL: http://dml.cz/dmlcz/700637

ON SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

J. SCHMETS

LIEGE

Let X be a completely regular and Hausdorff space, and E be a locally convex topological vector space.

Then we denote by $\mathscr{C}(X)$ the linear space of the continuous functions on X and by $C_s(X)$ the space $\mathscr{C}(X)$ endowed with the topology of pointwise or simple convergence.

As far as E is concerned, we denote by P its system of semi-norms and by P' a system of semi-norms on E which is finer than P.

If R is a locally convex property which is stable for Hausdorff inductive limits and which is satisfied by any linear space when it is equipped with its finest locally convex topology, then Y. Komura [2] has introduced the R-space associated to E as the linear space E endowed with the coarsest system of semi-norms P' which (is finer than P and which) makes E satisfy property R. As examples of such properties R let us mention the following ones : the space E is ultrabornological, bornological, barreled or evaluable.

Let us now recall the following results from [1].

Theorem 1.

a) The space $C_s(X)$ is always evaluable.

b) The barreled space associated to $C_s(X)$ is $C_c(\mu X)$. In particular, $C_s(X)$ is barreled if and only if X is a μ-space which compact subsets are finite.

c) The bornological space associated to $C_s(X)$ is $C_s(\upsilon X)$. In particular, $C_s(X)$ is bornological if and only if X is realcompact.

d) The ultrabornological space associated to $C_s(X)$ is $C_c(\upsilon X)$. In particular, $C_s(X)$ is ultrabornological if and only if X is realcompact and such that all its compact subsets are finite.

In this theorem, $C_c(X)$ stands of course for $\mathscr{C}(X)$ endowed with the compact-open topology coming from X, and υX for the realcompactification of X. Moreover, calling <u>bounding in</u> X the subsets of X where every $f \in \mathscr{C}(X)$ is bounded, μX is the smallest subspace of υX containing X and where every bounding subset is relatively compact.

<u>Definitions</u>. Let us denote by $\mathscr{C}(X;E)$ the linear space of the continuous functions from X into E and by $C_{P',s}(X;E)$ the space $\mathscr{C}(X;E)$ endowed with the semi-norms p'_A ($p' \in P'$, $A \subset X$ finite) defined by

$$p'_A(\varphi) = \sup_{x \in A} p'[\varphi(x)], \ \forall \varphi \in \mathscr{C}(X) .$$

Then $C_{P',s}(X;E)$ is of course a locally convex topological vector space, which we write $C_s(X;E)$ if $P' = P$.

We are interested in characterizing the spaces associated to $C_{P',s}(X;E)$. In [4], we have already got the following results.

<u>Theorem 2</u>.

a) The space $C_{P',s}(X;E)$ is a Mackey space if and only if (E,P') is a Mackey space.

b) The evaluable space associated to $C_{P',s}(X;E)$ is the space $C_{P'_e,s}(X;E)$ where P'_e is the system of semi-norms of the evaluable space associated to (E,P'). In particular, $C_{P',s}(X;E)$ is evaluable if and only if (E,P') is evaluable.

c) The space $C_{P',s}(X;E)$ is barreled if and only if $C_s(X)$ and (E,P') are barreled. Moreover, if $C_s(X)$ is barreled, then the barreled space associated to $C_{P',s}(X;E)$ is $C_{P'_t,s}(X;E)$ where P'_t is the system of semi-norms of the barreled space associated to (E,P').

The bornological and ultrabornological cases seem harder to study. In [4], we give some results about the bornological case. It is the purpose of this note to show that an analogous method allows to get results about the ultrabornological case. A more extended version is under preparation [5].

The hint goes on as follows. If B is an absolutely convex subset of E, then E_B denotes the linear hull of B when it is equipped with the gauge of B; moreover B is called a Banach disk if E_B is a Banach space. Then a sequence is Mackey (resp. fast) convergent in E to e if there is a bounded disk (resp. a bounded Banach disk) B of E such that the sequence converges to e in E_B. Finally E is bornological (resp. ultrabornological) if and only if every absolutely convex subset of E which absorbs the Mackey (resp. fast) convergent sequences is a neighborhood of 0 in E.

The following result indicates then where to look for.

Theorem 3. If the space $C_{P',s}(X;E)$ is ultrabornological (resp. bornological), then the spaces $C_s(X)$ and (E,P') are ultrabornological (resp. bornological).

The bornological result is theorem V.4.1 of [4]; the proof of the ultrabornological case goes on analogously by use of the preceding remark.

The aim is then to get converse results to theorem 3. This can be done with the help of the following generalization of a result of Nachbin [3].

Theorem 4. If T is an absolutely convex subset of $\mathscr{C}(X;E)$, there is a smallest compact subset K(T) of βX such that $\varphi \in \mathscr{C}(X;E)$ belongs to T if $\tilde{\varphi}$ is equal to 0 on a neighborhood of K(T) in βX ($\tilde{\varphi}$ represents the unique continuous extension of φ from βX into βE).

If moreover there are $p \in P$ and $r > 0$ such that

$$T \supset \{\varphi \in \mathscr{C}(X;E) : p[\varphi(x)] \leq r, \ \forall x \in X\} ,$$

then K(T) is the smallest compact subset of βX such that $\varphi \in \mathscr{C}(X;E)$ belongs to T if $\tilde{\varphi}$ vanishes on K(T), and one has then

$$T \supset \{\varphi \in \mathscr{C}(X;E) : \tilde{\varphi}(x) \in b_p(<r), \forall x \in K(T)\}.$$

<u>Proposition 5</u>. If T is an absolutely convex subset of $\mathscr{C}(X;E)$ which absorbs the fast convergent sequences of $C_{P',s}(X;E)$, then $K(T)$ is a subset of υX.

<u>Proof</u>. Suppose there is an element $x \in \beta X \setminus \upsilon X$ which belongs to $K(T)$. Then we know there are open subsets G_n of βX which are increasing, constitute a cover of υX and do not contain x. By theorem 4, there is then a sequence $\varphi_n \in \mathscr{C}(X;E)$ such that $\varphi_n \notin nD$ and $\tilde{\varphi}_n(G_n) = \{0\}$ for every n. One can prove then that

$$B = \{\sum_{n=1}^{\infty} c_n\, n\varphi_n : \sum_{n=1}^{\infty} |c_n| \leq 1\}$$

is an absolutely convex compact subset, hence a bounded Banach disk of $C_{P',s}(X;E)$. Of course the sequence φ_n tends to 0 in $\mathscr{C}(X;E)_B$ and cannot be absorbed by T, which is contradictory.

<u>Proposition 6</u>. If T is an absolutely convex subset of $\mathscr{C}(X;E)$ which absorbs the Mackey (resp. fast) convergent sequences of $C_{P',s}(X;E)$ [resp. of $C_{P,s}(X;E)$] and such that $K(T)$ is contained in X, and if X satisfies the first axiom of countability [resp. and if (E,P) is metrizable], then an element φ belongs to T if φ vanishes on $K(T)$.

<u>Proof</u>. Let us prove the "Mackey" version. By proposition V.4.7 of [4], we know that $K(T)$ is finite. So there is a sequence $f_n \in \mathscr{C}(X)$, with values in $[0,1]$, equal to 1 on neighborhoods of $K(T)$ and to 0 outside decreasing neighborhoods in X of $K(T)$, which intersection is $K(T)$. Let now $\varphi \in \mathscr{C}(X;E)$ vanish on $K(T)$. The absolutely convex hull of the sequence $n^2 f_n\varphi$ is then a bounded disk of $C_{P',s}(X;E)$: therefore the sequence $n f_n\varphi$ is absorbed by T. Moreover for every n, $[(1-f_n)\varphi]^\sim$ vanishes on a neighborhood in βX of $K(T)$, so $(1-f_n)\varphi$ belongs to εT for every $\varepsilon > 0$. Hence the conclusion since φ equals $\frac{1}{2}\,[(1-f_n)\varphi + f_n\varphi]$ for every n.

Let us now consider the "fast" version. Let φ vanish on $K(T)$ and $\{p_n : n \in \mathbb{N}\}$ be a countable system of semi-norms on E, equivalent to P. Then there is a sequence $f_n \in \mathscr{C}(X)$, with values in $[0,1]$, equal to 1 on a neighborhood of $K(T)$ and to 0 outside $G_n = \{x \in X : p_n[\varphi(x)] < n^{-4}\}$. Then

$$B = \{\sum_{n=1}^{\infty} c_n\, n^2\, f_n\varphi : \sum_{n=1}^{\infty} |c_n| \leq 1\}$$

is a bounded Banach disk of $C_{P,s}(X;E)$. But the sequence $nf_n\varphi$ converges to 0 in $\mathscr{C}(X;E)_B$, so it is absorbed by T. Hence the conclusion since $[(1-f_n)\varphi]^{\sim}$ vanishes on a neighborhood in βX of $K(T)$.

<u>Theorem 7</u>.

a) If $C_s(X)$ is bornological and (E,P) metrizable, then $C_s(X;E)$ is bornological.

b) If $C_s(X)$ is ultrabornological and (E,P) a Fréchet space, then $C_s(X;E)$ is ultrabornological.

<u>Proof</u>. a) is theorem V.4.11 of [4]. However one can proceed similarly to b), which simplifies the proof.

b) Let T be an absolutely convex subset of $\mathscr{C}(X;E)$ which absorbs the fast convergent sequences of $C_s(X;E)$. Combining theorem 4 and part d) of theorem 1, we get that $K(T)$ is a finite subset of X. Then there are $p \in P$ and $r > 0$ such that

$$T \supset \{\varphi \in \mathscr{C}(X;E) : p[\varphi(x)] \leq r,\ \forall x \in K(T)\} ;$$

by contradiction : if this is not the case, there is a sequence $\varphi_n \in \mathscr{C}(X;E)\setminus D$ such that $\sup_{x\in X} p_n[\varphi(x)] \leq n^{-4}$; then

$$B = \{\sum_{n=1}^{\infty} c_n\, n^2\, \varphi_n : \sum_{n=1}^{\infty} |c_n| \leq 1\}$$

is a bounded Banach disk of $C_s(X;E)$ and the sequence $n\varphi_n$ tends to 0 in $\mathscr{C}(X;E)_B$, which is contradictory to $\varphi_n \notin D$. The conclusion then follows from the last part of theorem 4.

Theorem 8. If $C_s(X)$ is bornological and if X satisfies the first axiom of countability, then the bornological space associated to $C_{P',s}(X;E)$ is the space $C_{P'_b,s}(X;E)$, where P'_b is the system of semi-norms of the bornological space associated to (E,P'). In particular, if $C_s(X)$ and (E,P') are bornological, then $C_{P',s}(X;E)$ is bornological.

Proof. This is theorem V.4.12 of [4] but one can proceed as in theorem 9 for the proof of the particular case, which simplifies considerably the developments used there.

Theorem 9. If $C_s(X)$ is ultrabornological and (E,P) a Fréchet space, and if X satisfies the first axiom of countability, then $C_s(X;E)$ is ultrabornological.

Proof. By part a) of theorem 2, $C_s(X;E)$ is already a Mackey space. To conclude, it is then sufficient to show that every linear functional $\mathcal{T}$ on $\mathscr{C}(X;E)$ which is bounded on the fast convergent sequences of $C_s(X;E)$ is continuous. But then

$$T = \{\varphi \in \mathscr{C}(X;E) \; : \; |\mathcal{T}(\varphi)| \leq 1\}$$

is absolutely convex in $\mathscr{C}(X;E)$ and absorbs the fast convergent sequences of $C_s(X;E)$. By theorems 4 and 5 and by part d) of theorem 1, $K(T)$ is a finite subset of X. Proposition 6 permits then to define a functional $\mathcal{Q}$ on the ultrabornological space $F = \prod_{x \in K(T)} (E,P)$ by

$$\mathcal{Q}(f) = \mathcal{T}(\varphi), \; \forall f \in F \; ,$$

if $\varphi \in \mathscr{C}(X;E)$ is such that $\varphi(x) = f(x)$ for every $x \in K(T)$. Then $\mathcal{Q}$ is a linear continuous functional on F since it is bounded on the fast convergent sequences of F. Therefore there are continuous linear functionals $\mathcal{T}_x$ on (E,P) for $x \in K(T)$ such that

$$\mathcal{T}(\varphi) = \sum_{x \in K(T)} \mathcal{T}_x [\varphi(x)], \; \forall \varphi \in \mathscr{C}(X;E) \; .$$

Hence the conclusion.

REFERENCES

[1] H. BUCHWALTER - J. SCHMETS, Sur le théorème de Nachbin-Shirota, J. Math. Pures et Appl., 52 (1973), 337-352.

[2] Y. KOMURA, On linear topological spaces, Kumamoto J. of Sc., 5A (1962), 148-157.

[3] L. NACHBIN, Topological vector spaces of continuous functions, Proc. Nat. Acad. USA, 40 (1954), 471-474.

[4] J. SCHMETS, Espaces de fonctions continues, Lecture Notes in Mathematics 519 (1976), Springer, Berlin-Heidelberg-New York.

[5] J. SCHMETS, Bornological and ultrabornological C(X;E) spaces, (in preparation).

[6] T. SHIROTA, On locally convex vector spaces of continuous functions, Proc. Japan Acad., 30 (1954), 294-298.

Université de Liège
Institut de Mathématique
Avenue des Tilleuls, 15

B-4000 LIEGE / BELGIUM